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Nonstationary Analogs of the Herglotz Representation Theorem: The Discrete Case

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For upper triangular operators with nonnegative real part, we derive generalized Herglotz representation theorems in which the main operator is coisometric, isometric, or unitary. The proofs are based on the representation theorems for upper triangular contractions considered earlier by D. Alpay and Y. Peretz. © 1999 Academic Press

1. INTRODUCTION

Time-invariant dissipative linear systems are characterized by Schur functions (that is, functions analytic and contractive in the open unit disk $\mathbb{D}$) or equivalently, by contractive Toeplitz upper triangular operators; see, for example, [24]. In the case of stochastic stationary processes, the functions which appear are the Carathéodory functions (that is, functions analytic in $\mathbb{D}$ with positive real part there). The associated reproducing kernel spaces of the kind introduced by de Branges and de Branges and Rovnyak play an important role, in setting as well as in solving problems

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such as interpolation problems [2, 3, 23], and the inverse scattering problem [6].

The passage from time-invariant systems and stationary processes to time-variant and general stochastic processes is of interest and has generated a number of approaches and theories; for instance, the theory of α-stationary processes introduced by Lev-Ari and Kailath [25] and the notion of displacement structures for matrices (see for instance [27]). Another approach originates with the work of E. Deprettere and P. Dewilde [20]. In that approach complex numbers are replaced by diagonal operators and multiplication by the independent complex variable by the bilateral shift in an ℓ_2-space. This approach was pursued and extended in [4, 21, 12]. An important feature of this generalization is the loss of commutativity, that is, the nonscalar diagonal operators do not commute with the bilateral shift.

All the problems in the stationary setting (that is, for Schur or Carathéodory functions) still make sense in the nonstationary setting. Some of these problems have already been considered: for instance the interpolation problems [21, 9] and the Adamyan–Arov–Krein theory, see [22], but a lot of problems remain to be solved, in particular in the framework of reproducing kernel spaces.

The purpose of this paper is to study realization results for bounded upper triangular operators with nonnegative real part (precise definitions will be given in Sections 2 and 3). These operators are the natural analogs of the Carathéodory functions. To set the problem into perspective, we first discuss the case of Carathéodory functions. Let ϕ be a $\mathbb{C}^{p\times p}$-valued function, analytic and with nonnegative real part in the open unit disk $\mathbb{D}$. Then the function

$$s(z) = (I_p - \phi(z))(I_p + \phi(z))^{-1}$$

is a Schur function. As is well known, the kernel

$$K_s(z, \omega) = \frac{I_p - s(z)\, s(\omega)^*}{1 - z\bar{\omega}}$$

is nonnegative in $\mathbb{D}$ (in the sense of reproducing kernels) and there is a uniquely defined reproducing kernel Hilbert space of $\mathbb{C}^p$-valued functions analytic in $\mathbb{D}$ with reproducing kernel K_s. This space is denoted by $\mathscr{H}(s)$ and is characterized by the following two properties:

(1) For every choice of $\omega \in \mathbb{D}$ and $\xi \in \mathbb{C}^p$, the function $z \mapsto K_s(z, \omega)\xi$ belongs to $\mathscr{H}(s)$.

(2) For every $\omega \in \mathbb{D}$, $\xi \in \mathbb{C}^p$, and $f \in \mathscr{H}(s)$, we have $\langle f, K_s(\cdot, \omega)\xi \rangle_{\mathscr{H}(s)} = \xi^* f(\omega)$, where $\langle \cdot, \cdot \rangle_{\mathscr{H}(s)}$ denotes the inner product in $\mathscr{H}(s)$.

The space $\mathscr{H}(s)$ is the state space for a coisometric realization of s, namely

$$s(z)\xi = (D_s + zC_s(I_{\mathscr{H}(S)} - zA_s)^{-1} B_s)(\xi), \qquad z \in \mathbb{D}, \tag{1.1}$$

where the operators A_s, B_s, C_s, and D_s are bounded and defined by

$$\begin{aligned}(A_s f)(z) &= \frac{f(z)-f(0)}{z}\\ (B_s\xi)(z) &= \frac{s(z)-s(0)}{z}\,\xi\\ C_s(f) &= f(0)\\ D_s(\xi) &= s(0)\xi\end{aligned} \tag{1.2}$$

and the operator matrix

$$\begin{pmatrix} A_s & B_s \\ C_s & D_s \end{pmatrix} : \begin{pmatrix} \mathscr{H}(s) \\ \mathbb{C}^p \end{pmatrix} \to \begin{pmatrix} \mathscr{H}(s) \\ \mathbb{C}^p \end{pmatrix}$$

is coisometric. We shall also use the term *colligation* to designate such quadruples of operators. Using the expression of ϕ in terms of s, one sees that the function

$$K_\phi(z, \omega) = \frac{\phi(z) + \phi(\omega)^*}{1 - z\bar{\omega}} = \frac{(I_p + \phi(z))}{\sqrt{2}} \, \frac{I_p - s(z)\, s(\omega)^*}{1 - z\bar{\omega}} \, \frac{(I_p + \phi(\omega)^*)}{\sqrt{2}}$$

is nonnegative on $\mathbb{D} \times \mathbb{D}$. Hence it generates a uniquely determined Hilbert space $\mathscr{L}_+(\phi)$ with reproducing kernel K_ϕ. The mapping $f(z) \in \mathscr{H}(s) \mapsto ((I_p + \phi(z))/\sqrt{2})\, f(z) \in \mathscr{L}_+(\phi)$ is unitary and the formulas

$$\begin{aligned}(A_\phi f)(z) &= \frac{f(z)-f(0)}{z}\\ (B_\phi\xi)(z) &= \frac{\phi(z)-\phi(0)}{z}\,\xi\\ C_\phi(f) &= f(0)\\ D_\phi(\xi) &= \phi(0)\xi\end{aligned} \tag{1.3}$$

define an operator matrix

$$\begin{pmatrix} A_\phi & B_\phi \\ C_\phi & D_\phi \end{pmatrix} : \begin{pmatrix} \mathscr{L}_+(\phi) \\ \mathbb{C}^p \end{pmatrix} \to \begin{pmatrix} \mathscr{L}_+(\phi) \\ \mathbb{C}^p \end{pmatrix}$$

such that

$$\phi(z)\xi = (D_\phi + zC_\phi(I_{\mathscr{L}_+(\phi)} - zA_\phi)^{-1} B_\phi)(\xi), \qquad z \in \mathbb{D}. \tag{1.4}$$

For this, and more generally for the theory of $\mathscr{H}(s)$ and $\mathscr{L}_+(\phi)$ spaces, we refer to the works [14, 15] of de Branges and [17] of de Branges and Rovnyak. Note that (1.4) can also be written as

$$\phi(z)\xi = (i \operatorname{Im} D_\phi + \tfrac{1}{2} C_\phi (I + zA_\phi)(I - zA_\phi)^{-1} C_\phi^*)(\xi), \qquad z \in \mathbb{D}. \tag{1.5}$$

This follows from $(C_\phi^* \xi)(z) = K_\phi(z, 0)\xi = (\phi(z) + \phi(0)^*)\xi$, which implies

$$\tfrac{1}{2} C_\phi C_\phi^*(\xi) = \tfrac{1}{2}(\phi(0) + \phi(0)^*)\xi = (\operatorname{Re} D_\phi)(\xi),$$

and from

$$A_\phi C_\phi^*(\xi) = \frac{\phi(z) - \phi(0)}{z}\, \xi = B_\phi(\xi).$$

The main operator A_ϕ is coisometric since for $\omega \neq 0$,

$$A_\phi^*(K_\phi(\cdot, \omega)\xi)(z) = \frac{K_\phi(z, \omega) - K_\phi(z, 0)}{\bar{\omega}}\, \xi,$$

$$A_\phi A_\phi^*(K_\phi(\cdot, \omega)\xi)(z) = K_\phi(z, \omega)\xi,$$

and hence $A_\phi A_\phi^* = I_{\mathscr{L}_+(\phi)}$. Realizations similar to (1.5) with isometric or unitary A_ϕ are also possible. To obtain them one extends ϕ to the complement $\mathbb{E}$ of the closed unit disk by $\phi(z) = -\phi(1/z^*)^*$ and considers the kernel K_ϕ with $z, \omega \in \mathbb{E}$ and $\mathbb{D} \cup \mathbb{E}$ respectively. To get a unitary A_ϕ one can also consider a suitable 2×2 matrix kernel, and this is what we will do in this paper for upper triangular operators. If A_ϕ in (1.5) is unitary, then one can derive an integral representation for ϕ. Indeed, let $\{E(\lambda)\}_{t \in \mathbb{R}}$ with support $[0, 2\pi]$ be the resolution of the identity of A_ϕ, that is,

$$A_\phi = \int_0^{2\pi} e^{i\lambda}\, dE(\lambda),$$

then

$$\phi(z)\xi = \left(i \operatorname{Im} D_\phi + \int_0^{2\pi} \frac{1 + ze^{i\lambda}}{1 - ze^{i\lambda}}\, d\mu(e^{i\lambda}) \right)(\xi),$$

where $d\mu(e^{i\lambda}) = \frac{1}{2} C_\phi\, dE(\lambda)\, C_\phi^*$ is a finite nonnegative $\mathbb{C}^{p \times p}$-valued Borel measure on the unit circle $\mathbb{T}$. As is well known, a converse construction is also possible, that is, from the integral representation one can get (1.5); see, for example, [1, 18].

In this paper we prove realization formulas similar to (1.5) for upper triangular operators with nonnegative real part; see Theorems 4.4, 5.2, and 6.6. It was suggested by Ph. Loubaton that using the formula with unitary main operator one can associate in a natural way an operator-valued measure to a nonstationary stochastic process; this will be considered in the last section. To prove the representation formulas, we use the corresponding results for upper triangular contractions in the same way as we explained above for the matrix-valued function ϕ. The realization theory for upper triangular contractions was developed in [7] and based on reproducing kernel methods; see also [19] for more algebraic methods.

The outline of the paper is as follows: the paper consists of six sections besides this Introduction. In Section 2 we review the nonstationary setting which was developed in [5, 21]. The analog of the $\mathscr{L}_+(\phi)$ spaces for upper triangular operators is studied in Section 3. The next three sections are devoted to Herglotz-type formulas in which the main operator is coisometric, isometric and unitary. In the last section we show that the entries of a Carathéodory operator can be represented in terms of a nonnegative operator-valued Borel measure.

2. THE NONSTATIONARY SETTING

Let $\mathscr{N}$ be a separable Hilbert space, "the coefficient space," and denote by $\ell^2_{\mathscr{N}}$ the Hilbert space of all two sided square summable sequences $f=(f)_{i=-\infty}^{\infty}=(..., f_{-1}, \boxed{f_0}, f_1, ...)^t$ with components $f_i \in \mathscr{N}$ provided with the standard inner product. As in [21, Section 1] the set of bounded linear operators from $\ell^2_{\mathscr{N}}$ into itself is denoted by $\mathscr{X}$. Let Z denote the bilateral backward shift operator

$$(Zf)_i = f_{i+1}, \qquad i = ..., -1, 0, 1, ...$$

It is unitary on $\ell^2_{\mathscr{N}}$, that is, $ZZ^* = Z^*Z = I$, and

$$\pi^* Z^j \pi = \begin{cases} I & \text{if } j = 0, \\ 0 & \text{if } j \neq 0, \end{cases}$$

where π denotes the injection map: $u \in \mathscr{N} \mapsto (..., 0, \boxed{u}, 0, ...)^t \in \ell^2_{\mathscr{N}}$.

An element $A \in \mathscr{X}$ can be represented as an operator matrix (A_{ij}) with $A_{ij} = \pi^* Z^i A Z^{*j} \pi$. The spaces of upper triangular, lower triangular, and diagonal operators will be denoted by $\mathscr{U}$, $\mathscr{L}$, and $\mathscr{D}$:

$$\mathscr{U} = \{A \in \mathscr{X} \mid A_{ij} = 0, i > j\}, \qquad \mathscr{L} = \{A \in \mathscr{X} \mid A_{ij} = 0, j > i\}, \qquad \mathscr{D} = \mathscr{U} \cap \mathscr{L}.$$

A contraction in $\mathscr{X}$ will be called a Schur operator.

Note that the columns of AZ and AZ^* are obtained by shifting those of A one place to the east and west respectively; the rows of ZA and Z^*A are obtained by shifting those of A one place to the north and south respectively. Let $A^{(j)} = Z^{*j}AZ^j$ for $A \in \mathscr{X}$ and $j = ..., -1, 0, 1, ...$; then the entries of $A^{(j)}$ are obtained by shifting the entries of A in the south-east direction j places: $(A^{(j)})_{st} = A_{s-j,\, t-j}$. Clearly the map $A \mapsto A^{(j)}$ takes the spaces $\mathscr{U}$, $\mathscr{L}$, $\mathscr{D}$ into themselves, $A^{(j+k)} = (A^{(j)})^{(k)}$, and $(AB)^{(j)} = A^{(j)}B^{(j)}$.

For $W \in \mathscr{X}$ we define

$$W^{[0]} = I, \qquad W^{[n]} = WW^{(1)}W^{(2)} \cdots W^{(n-1)} = (WZ^*)^n Z^n, \qquad n \geqslant 1.$$

For any $F \in \mathscr{U}$, there exists a unique sequence of operators $F_{[j]} \in \mathscr{D}$, $j = 0, 1, ...$, namely $(F_{[j]})_{ii} = F_{i-j,\, i}$, such that

$$F = \sum_{n=0}^{\infty} Z^n F_{[n]} \tag{2.1}$$

in the sense that $F - \sum_{j=0}^{n-1} Z^j F_{[j]} \in Z^n \mathscr{U}$, and also in a weak sense:

LEMMA 2.1. *If $F \in \mathscr{U}$, then the series $\sum_{n=0}^{\infty} F_{\{n\}} Z^n$ converges weakly to F, that is, for every $h \in \ell^2_{\mathscr{N}}$ and every $g \in \ell^2_{\mathscr{M}}$,*

$$\langle Fh, g \rangle_{\ell^2_{\mathscr{M}}} = \left\langle \sum_{n=0}^{\infty} F_{\{n\}} Z^n h, g \right\rangle_{\ell^2_{\mathscr{M}}}.$$

Proof. The identity follows from the computation

$$\begin{aligned}
\langle Fh, g \rangle_{\ell^2_{\mathscr{M}}} &= \operatorname{Tr} g^* F h \\
&= \operatorname{Tr} \sum_{k=-\infty}^{\infty} \sum_{j=k}^{\infty} g_k^* F_{k,\, j} h_j \\
&= \operatorname{Tr} \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} g_k^* F_{k,\, k+n} h_{k+n} \\
&= \operatorname{Tr} \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} g_k^* (F_{\{n\}})_{k,\, k}\, h_{k+n} \\
&= \operatorname{Tr} \sum_{n=0}^{\infty} g^* F_{\{n\}} Z^n h \\
&= \left\langle \sum_{n=0}^{\infty} F_{\{n\}} Z^n h, g \right\rangle_{\ell^2_{\mathscr{M}}}. \qquad \blacksquare
\end{aligned}$$

One can view the series (2.1) as a formal "left" powers series in Z and we define the left point evaluation of this power series at $W \in \mathcal{X}$ by (note that $Z^n = Z^{[n]}$)

$$F^{\wedge}(W) = \sum_{n=0}^{\infty} W^{[n]} F_{[n]}.$$

Similarly, there exists a unique sequence of diagonal operators $F_{\{j\}} \in \mathcal{D}$, namely $(F_{\{j\}})_{ii} = F_{i,\, i+j}$ (so $F_{\{j\}} = Z^j F_{[j]} Z^{*j} = F_{[j]}^{(-j)}$), such that

$$F = \sum_{n=0}^{\infty} F_{\{n\}} Z^n$$

in the sense that $F - \sum_{j=0}^{n-1} F_{\{j\}} Z^j \in \mathcal{U} Z^n$, or in the weak sense as in Lemma 2.1. This series is a formal "right" power series in Z. For $W \in \mathcal{X}$ we set

$$W^{\{n\}} = Z^n (Z^* W)^n, \qquad n = 0, 1, \ldots$$

and we define the right point evaluation of F at $W \in \mathcal{X}$ by

$$F^{\triangle}(W) = \sum_{n=0}^{\infty} F_{\{n\}} W^{\{n\}}.$$

For $W \in \mathcal{X}$, it holds that

$$\ell_W = \lim_{n \to \infty} \| W^{[n]} \|^{1/n} = \lim_{n \to \infty} \| W^{\{n\}} \|^{1/n} = r_{sp}(WZ^*) = r_{sp}(Z^* W),$$

where $r_{sp}(V)$ stands for the spectral radius of V. If $\ell_W < 1$ then the series defining the left and right point evaluations converge in the uniform operator norm of $\mathcal{X}$. This holds in particular for the operators W in

$$\Omega = \{ W \in \mathcal{D} \mid \ell_W < 1 \}.$$

The set Ω contains in particular the diagonal elements in $\mathcal{X}$ of norm strictly less than 1 and is the nonstationary analog of the open unit disk.

In the sequel we will be mostly concerned with the maps $W \mapsto F^{\wedge}(W)$ and $W \mapsto F^{\triangle}(W)$ for $W \in \mathcal{D}$; then the operators $F^{\wedge}(W)$ and $F^{\triangle}(W)$ are also diagonal. The next theorem from [5] characterizes $F^{\wedge}(W)$ and $F^{\triangle}(W)$ in a different way.

THEOREM 2.2. *Let $F \in \mathcal{U}$ and $D \in \mathcal{D}$. (1) The operator $(Z - W)^{-1}(F - D)$ belongs to $\mathcal{U}$ for $W \in \Omega$ if and only if $D = F^{\wedge}(W)$. (2) The operator $(F - D)(Z - W)^{-1}$ belongs to $\mathcal{U}$ for $W \in \Omega$ if and only if $D = F^{\triangle}(W)$.*

Continuous analogs of these generalized point evaluations are introduced in [10, 11]. We will use them in a forthcoming publication to get the continuous analogs of the results of the present paper.

An operator $F=(F_{ij})_{i,j}\in\mathcal{X}$ is a Hilbert–Schmidt operator if all its entries F_{ij} are Hilbert–Schmidt operators on $\mathcal{N}$ and $\sum_{i,j}\operatorname{Tr} F_{ij}^*F_{ij}<\infty$, where Tr stands for trace. The set of these operators will be denoted by $\mathcal{X}_2$ and it is a Hilbert space with respect to the inner product

$$\langle F, G\rangle_{\mathcal{X}_2}=\sum_{i,j}\operatorname{Tr} G_{ij}^*F_{ij}<\infty.$$

The subspaces of upper triangular, lower triangular and diagonal operators in $\mathcal{X}_2$ will be denoted by $\mathcal{U}_2$, $\mathcal{L}_2$ and $\mathcal{D}_2$. The space $\mathcal{U}_2$ is a reproducing kernel Hilbert space with reproducing kernel

$$\rho_W^{\wedge\,-1}=(I-ZW^*)^{-1}=\sum_0^\infty(ZW^*)^n=\sum_0^\infty Z^nW^{[n]*}$$

in the sense that for all $W\in\Omega$, $E\in\mathcal{D}_2$, and $F\in\mathcal{U}_2$, the operator $\rho_W^{\wedge\,-1}E\in\mathcal{U}_2$ and

$$\langle F,\rho_W^{\wedge\,-1}E\rangle_{\mathcal{U}_2}=\operatorname{Tr} E^*F^\wedge(W).$$

For $X\in\mathcal{U}$, denote by $\mathcal{M}_X^\ell$ the left multiplication operator $F\mapsto XF$ from $\mathcal{U}_2$ to itself. Then by the reproducing kernel formula,

$$\mathcal{M}_X^{\ell*}(\rho_W^{\wedge\,-1}E)=X^\wedge(W)^*\,\rho_W^{\wedge\,-1}E,\qquad E\in\mathcal{D}_2. \tag{2.2}$$

PROPOSITION 2.3. *Let $\Gamma\colon\mathcal{U}_2\to\mathcal{U}_2$ be a bounded nonnegative operator and let P be the orthogonal projection onto* Ker Γ. *Then the operator range* Ran $\Gamma^{1/2}$ *provided with the lifted norm*

$$\|\Gamma^{1/2}G\|_{\operatorname{Ran}\Gamma^{1/2}}=\|(I-P)\,G\|_{\mathcal{U}_2},\qquad G\in\mathcal{U}_2,$$

is a reproducing kernel Hilbert space with reproducing kernel $\Gamma\rho_W^{\wedge\,-1}$ in the sense that for all $W\in\Omega$, $E\in\mathcal{D}_2$, and $F\in$ Ran $\Gamma^{1/2}$, *the operator $\Gamma(\rho_W^{\wedge\,-1}E)$ belongs to* Ran $\Gamma^{1/2}$ *and*

$$\langle F,\Gamma(\rho_W^{\wedge\,-1}E)\rangle_{\operatorname{Ran}\Gamma^{1/2}}=\operatorname{Tr} E^*F^\wedge(W). \tag{2.3}$$

Proof. We only prove the reproducing kernel property. For $F\in\operatorname{Ran}\Gamma^{1/2}$, $F=\Gamma^{1/2}G$ with $G\in\mathcal{U}_2$,

$$\begin{aligned}\langle F,\Gamma(\rho_W^{\wedge\,-1}E)\rangle_{\operatorname{Ran}\Gamma^{1/2}}&=\langle G,\Gamma^{1/2}(\rho_W^{\wedge\,-1}E)\rangle_{\mathcal{U}_2}\\&=\langle F,\rho_W^{\wedge\,-1}E\rangle_{\mathcal{U}_2}=\operatorname{Tr} E^*F^\wedge(W).\quad\blacksquare\end{aligned}$$

It is well known that convergence in norm in a reproducing kernel Hilbert space implies pointwise convergence. The analog of this fact in the present setting follows from (2.3):

COROLLARY 2.4. *If in the operator range in the previous proposition* F_n *is a sequence converging to* F, *then for every* $W \in \Omega$ *and* $E \in \mathscr{D}_2$, $\operatorname{Tr} E^* F_n^\wedge(W)$ *converges to* $\operatorname{Tr} E^* F^\wedge(W)$.

For more about reproducing kernel Hilbert spaces and operator ranges in this context, see [5, Sects. 7 and 8].

3. THE SPACES $\mathscr{H}_\ell(S)$ AND $\mathscr{L}_\ell(\Phi)$ IN THE NONSTATIONARY CASE

In the sequel Φ is called a Carathéodory operator if $\Phi \in \mathscr{U}$ and Φ has a nonnegative real part, that is, $\operatorname{Re} \Phi = (\Phi + \Phi^*)/2 \geqslant 0$. Then the left multiplication operator $\mathscr{M}_\Phi^\ell \colon \mathscr{U}_2 \to \mathscr{U}_2$ defined by $\mathscr{M}_\Phi^\ell(F) = \Phi F$ is well defined and bounded with $\|\mathscr{M}_\Phi^\ell\| \leqslant \|\Phi\|$, the latter follows from $\|\Phi F\|_{\mathscr{U}_2} \leqslant \|\Phi\| \, \|F\|_{\mathscr{U}_2}$.

LEMMA 3.1. *Let* Φ *be a Carathéodory operator. The operator* $(I+\Phi)$ *is boundedly invertible from* $\ell_{\mathscr{N}}^2$ *into itself and the operator* $(I+\Phi)^{-1}$ *belongs to* $\mathscr{U}$.

Proof. For $f \in \ell_{\mathscr{N}}^2$,

$$\|(I+\Phi) f\|_{\ell_{\mathscr{N}}^2}^2 = \langle f, f\rangle_{\ell_{\mathscr{N}}^2} + 2\langle \operatorname{Re} \Phi f, f\rangle_{\ell_{\mathscr{N}}^2} + \langle \Phi f, \Phi f\rangle_{\ell_{\mathscr{N}}^2} \geqslant \|f\|_{\ell_{\mathscr{N}}^2}^2.$$

Similarly, $\|(I+\Phi^*) f\|_{\ell_{\mathscr{N}}^2} \geqslant \|f\|_{\ell_{\mathscr{N}}^2}$. Hence, by the closed graph theorem, $(I+\Phi)$ is boundedly invertible from $\ell_{\mathscr{N}}^2$ into itself. Now define $\mathscr{M}_{(I+\Phi)}^\ell \colon \mathscr{X}_2 \to \mathscr{X}_2$ by $M_{(I+\Phi)}^\ell(F) = (I+\Phi) F$. Then $\mathscr{M}_{(I+\Phi)}^\ell$ is boundedly invertible, and $(\mathscr{M}_{(I+\Phi)}^\ell)^{-1} = \mathscr{M}_{(I+\Phi)^{-1}}^\ell$.

For $F \in \mathscr{U}_2$ we write $\mathscr{M}_{(I+\Phi)^{-1}}^\ell(F) = (I+\Phi)^{-1} F = G + H$ where $G \in \mathscr{U}_2$ and $H \in Z^* \mathscr{L}_2$. Then $F = (I+\Phi) G + (I+\Phi) H$, $(I+\Phi) H \in \mathscr{U}_2$ and $\langle (I+\Phi) H, H\rangle_{\mathscr{X}_2} = 0$. The latter equality implies that

$$\langle H, H\rangle_{\mathscr{X}_2} + \langle (\operatorname{Re} \Phi) H, H\rangle_{\mathscr{X}_2} = 0.$$

Since both terms are nonnegative, $H = 0$. Therefore $(I+\Phi)^{-1} F \in \mathscr{U}_2$, and since $F \in \mathscr{U}_2$ is arbitrary, $(I+\Phi)^{-1} \in \mathscr{U}$. ∎

There is a natural contraction $S \in \mathcal{U}$ associated to a Carathéodory operator Φ, namely $S = (I+\Phi)^{-1}(I-\Phi) = (I-\Phi)(I+\Phi)^{-1}$. Indeed, by Lemma 3.1 $S \in \mathcal{U}$, and since

$$I - SS^* = (I+\Phi)^{-1}\{(I+\Phi)(I+\Phi^*) - (I-\Phi)(I-\Phi^*)\}(I+\Phi^*)^{-1}$$
$$= 4(I+\Phi)^{-1}\{\operatorname{Re}\Phi\}(I+\Phi^*)^{-1} \geqslant 0$$

S is a Schur operator. Conversely,

$$I + S = (I+\Phi)^{-1}\{(I+\Phi)+(I-\Phi)\} = 2(I+\Phi)^{-1},$$

hence $I+S$ is invertible in $\mathcal{U}$ and $\Phi = (I+S)^{-1}(I-S) = (I-S)(I+S)^{-1}$.

For any Schur function $S \in \mathcal{U}$ the operator of left multiplication by S, $\mathcal{M}^{\ell}_S(F) = SF$ is a contraction from $\mathcal{U}_2$ into itself. By $\mathcal{H}_\ell(S) \subset \mathcal{U}_2$ we denote the operator range

$$\mathcal{H}_\ell(S) = \operatorname{Ran}(I_{\mathcal{U}_2} - \mathcal{M}^{\ell}_S \mathcal{M}^{\ell *}_S)^{1/2}$$

with the lifted norm

$$\|(I_{\mathcal{U}_2} - \mathcal{M}^{\ell}_S \mathcal{M}^{\ell *}_S)^{1/2} F\|_{\mathcal{H}_\ell(S)} = \|(I-P)F\|_{\mathcal{U}_2},$$

where P is the orthogonal projection in $\mathcal{U}_2$ onto $\operatorname{Ker}(I_{\mathcal{U}_2} - \mathcal{M}^{\ell}_S \mathcal{M}^{\ell *}_S)$. By de Brange's complementation theory (see [16; 8, Theorem 3.9]), this space can be characterized as

$$\mathcal{H}_\ell(S) = \{F \in \mathcal{U}_2 \mid \kappa(F) < \infty\}, \qquad \kappa(F) = \sup_{G \in \mathcal{U}_2} \{\|F + SG\|^2_{\mathcal{U}_2} - \|G\|^2_{\mathcal{U}_2}\},$$

and $\kappa(F) = \|F\|^2_{\mathcal{H}_\ell(S)}$, $F \in \mathcal{H}_\ell(S)$. In particular, $\|F\|_{\mathcal{U}_2} \leqslant \|F\|_{\mathcal{H}_\ell(S)}$, that is, $\mathcal{H}_\ell(S)$ is contractively included in $\mathcal{U}_2$.

In a similar way we define the operator range

$$\mathcal{L}_\ell(\Phi) = \operatorname{Ran}(\mathcal{M}^{\ell}_\Phi + \mathcal{M}^{\ell *}_\Phi)^{1/2} \subset \mathcal{U}_2;$$

it is a Hilbert space when equipped with the lifted norm

$$\|(\mathcal{M}^{\ell}_\Phi + \mathcal{M}^{\ell *}_\Phi)^{1/2} F\|_{\mathcal{L}_\Phi(S)} = \|(I-Q)F\|_{\mathcal{U}_2},$$

where Q is the orthogonal projection in $\mathcal{U}_2$ onto $\operatorname{Ker}(\mathcal{M}^{\ell}_\Phi + \mathcal{M}^{\ell *}_\Phi)$. By Proposition 2.3, $\mathcal{H}_\ell(S)$ and $\mathcal{L}_\ell(\Phi)$ are reproducing kernel Hilbert spaces with reproducing kernels $\Gamma_S(\rho_W^{\wedge\,-1} E)$ and $\Gamma_\Phi(\rho_W^{\wedge\,-1} E)$, $W \in \Omega$, $E \in \mathcal{D}_2$ where

$$\Gamma_S = I_{\mathcal{U}_2} - \mathcal{M}^{\ell}_S \mathcal{M}^{\ell *}_\Phi, \qquad \Gamma_\Phi = \mathcal{M}^{\ell}_\Phi + \mathcal{M}^{\ell *}_\Phi.$$

These two spaces are related in the following way:

LEMMA 3.2. *Let Φ be a Carathéodory operator and $S=(I+\Phi)^{-1}(I-\Phi)$ be the associated Schur function. Then $\mathcal{L}_\ell(\Phi)=(1/\sqrt{2})(I+\Phi)\mathcal{H}_\ell(S)$, and the multiplication operator $\mathcal{M}^\ell_{(1/\sqrt{2})(I+\Phi)}:\mathcal{H}_\ell(S)\to\mathcal{L}_\ell(\Phi)$ is unitary.*

Proof. Since $\mathcal{M}^\ell_S=\mathcal{M}^\ell_{(I+\Phi)^{-1}}\mathcal{M}^\ell_{(I-\Phi)}$ we have

$$\begin{aligned}\Gamma_S&=\mathcal{M}^\ell_{(I+\Phi)^{-1}}\{\mathcal{M}^\ell_{(I+\Phi)}\mathcal{M}^{\ell*}_{(I+\Phi)}-\mathcal{M}^\ell_{(I-\Phi)}\mathcal{M}^{\ell*}_{(I-\Phi)}\}\mathcal{M}^{\ell*}_{(I+\Phi)^{-1}}\\&=2\mathcal{M}^\ell_{(I+\Phi)^{-1}}\Gamma_\Phi\mathcal{M}^{\ell*}_{(I+\Phi)^{-1}},\end{aligned}$$

and hence

$$\Gamma_\Phi=\mathcal{M}^\ell_{(1/\sqrt{2})(I+\Phi)}\Gamma_S\mathcal{M}^{\ell*}_{(1/\sqrt{2})(I+\Phi)}. \tag{3.1}$$

Let $\mathbf{R}\in\mathcal{H}_\ell(S)\times\mathcal{L}_\ell(\Phi)$ be the linear relation spanned by elements of the form

$$(\Gamma_S\mathcal{M}^{\ell*}_{(1/\sqrt{2})(I+\Phi)}(\rho^{\wedge -1}_W E),\ \Gamma_\Phi(\rho^{\wedge -1}_W E)),\qquad W\in\Omega,\qquad E\in\mathcal{D}_2.$$

We claim that $\mathbf{R}$ is isometric and has dense domain Dom $\mathbf{R}$ and dense range Ran $\mathbf{R}$. Before proving this we proceed with the argument. The claim implies that the closure of $\mathbf{R}$ is the graph of a unitary operator U from $\mathcal{H}_\ell(S)$ onto $\mathcal{L}_\ell(\Phi)$. From the definition of $\mathbf{R}$ and (3.1) U coincides with $\mathcal{M}^\ell_{(1/\sqrt{2})(I+\Phi)}$ on Dom $\mathbf{R}$. We prove that $U=\mathcal{M}^\ell_{(1/\sqrt{2})(I+\Phi)}$ on the whole space $\mathcal{H}_\ell(S)$. Let $F\in\mathcal{H}_\ell(S)$ and let F_n be a sequence in Dom $\mathbf{R}$ which converges to F. Since $\mathcal{H}_\ell(S)$ is contractively included in $\mathcal{U}_2$, F_n converges to also F in $\mathcal{U}_2$. The operator $\mathcal{M}^\ell_{(1/\sqrt{2})(I+\Phi)}$ is bounded in $\mathcal{U}_2$ and hence

$$\operatorname{Tr} E^*(\mathcal{M}^\ell_{(1/\sqrt{2})(I+\Phi)}(F_n))^\wedge(W)\xrightarrow{n\to\infty}\operatorname{Tr} E^*(\mathcal{M}^\ell_{(1/\sqrt{2})(I+\Phi)}(F))^\wedge(W). \tag{3.2}$$

The sequence on the left coincides with the sequence $\operatorname{Tr} E^*(U(F_n))^\wedge(W)$ which converges to $\operatorname{Tr} E^*(U(F))^\wedge(W)$. It follows that

$$\operatorname{Tr} E^*(\mathcal{M}^\ell_{(1/\sqrt{2})(I+\Phi)}(F))^\wedge(W)=\operatorname{Tr} E^*(U(F))^\wedge(W)$$

for all $W\in\Omega$ and $E\in\mathcal{D}_2$. This implies that $U(F)=\mathcal{M}^\ell_{(1/\sqrt{2})(I+\Phi)}(F)$ for all $F\in\mathcal{H}_\ell(S)$.

It remains to prove the three parts of the claim. First we show that $\mathbf{R}$ is isometric:

$$\left\| \sum_i \Gamma_S \mathcal{M}^{\ell *}_{(1/\sqrt{2})(I+\Phi)}(\rho_{\hat{W}_i}^{-1} E_i) \right\|^2_{\mathcal{H}_\ell(S)}$$

$$= \sum_{i,j} \langle \Gamma_S \mathcal{M}^{\ell *}_{(1/\sqrt{2})(I+\Phi)}(\rho_{\hat{W}_i}^{-1} E_i), \mathcal{M}^{\ell *}_{(1/\sqrt{2})(I+\Phi)}(\rho_{\hat{W}_j}^{-1} E_j) \rangle_{\mathcal{U}_2}$$

$$= \sum_{i,j} \langle \Gamma_\Phi(\rho_{\hat{W}_i}^{-1} E_i), \rho_{\hat{W}_j}^{-1} E_j \rangle_{\mathcal{U}_2}$$

$$= \left\| \sum_i \Gamma_\Phi(\rho_{\hat{W}_i}^{-1} E_i) \right\|^2_{\mathcal{L}_\ell(\Phi)}.$$

To show that $\operatorname{Dom} \mathbf{R}$ is dense, consider $F \in \mathcal{H}_\ell(S) \ominus \operatorname{Dom} \mathbf{R}$. Then $F = \Gamma_S^{1/2}(G)$ for some $G \in \mathcal{U}_2$, and for every $W \in \Omega$ and $E \in \mathcal{D}_2$,

$$0 = \langle F, \Gamma_S \mathcal{M}^{\ell *}_{(1/\sqrt{2})(I+\Phi)}(\rho_{\hat{W}}^{-1} E) \rangle_{\mathcal{H}_\ell(S)}$$

$$= \langle G, \Gamma_S^{1/2} \mathcal{M}^{\ell *}_{(1/\sqrt{2})(I+\Phi)}(\rho_{\hat{W}}^{-1} E) \rangle_{\mathcal{U}_2}$$

$$= \langle F, \mathcal{M}^{\ell *}_{(1/\sqrt{2})(I+\Phi)}(\rho_{\hat{W}}^{-1} E) \rangle_{\mathcal{U}_2} = \left\langle \frac{1}{\sqrt{2}} (I+\Phi) F, \rho_{\hat{W}}^{-1} E \right\rangle_{\mathcal{U}_2},$$

hence $(1/\sqrt{2})(I+\Phi) F = 0$, that is, $F = 0$. Finally, $\operatorname{Ran} \mathbf{R}$ is dense by Proposition 2.3. ∎

4. THE COISOMETRIC REPRESENTATION

The first main result of this paper is Theorem 4.4 below. It is a non-stationary analog of the Herglotz representation formula (1.5) with coisometric main operator. The starting point of this section is [7, Theorems 4.1, 4.5, and 4.6].

THEOREM 4.1. *Let $S \in \mathcal{U}$ be a Schur operator. Then for all $E \in \mathcal{D}_2$ and $W \in \mathcal{D}$ with $\|W\| < 1$,*

$$(SE)^{\triangle}(W) = (\mathbf{D}_S + \mathbf{C}_S \mathcal{M}^r_W (\mathbf{I}_{\mathcal{H}_\ell(S)} - \mathbf{A}_S \mathcal{M}^r_W)^{-1} \mathbf{B}_S)(E),$$

where $\mathscr{M}^r_W$ denotes the operator of right multiplication by W in $\mathscr{H}_\ell(S)$,

$$\begin{aligned}
\mathbf{A}_S(F) &= (F - F_{[0]})\, Z^{-1}\\
\mathbf{B}_S(E) &= (S - S_{[0]})\, EZ^{-1}\\
\mathbf{C}_S(F) &= F_{[0]}\\
\mathbf{D}_S(E) &= S_{[0]}E
\end{aligned} \tag{4.1}$$

define bounded operators and the colligation

$$\mathscr{V}_S = \begin{pmatrix} \mathbf{A}_S & \mathbf{B}_S \\ \mathbf{C}_S & \mathbf{D}_S \end{pmatrix} : \begin{pmatrix} \mathscr{H}_\ell(S) \\ \mathscr{D}_2 \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_\ell(S) \\ \mathscr{D}_2 \end{pmatrix} \tag{4.2}$$

is coisometric: $\mathscr{V}_S \mathscr{V}_S^* = I_{\mathscr{H}_\ell(S)}$, and closely outer-connected:

$$\mathscr{H}_\ell(S) = \overline{\operatorname{span}}\{ \operatorname{Ran}(\mathbf{A}_S^*)^n\, \mathbf{C}_S^* \mid n \geqslant 0 \}.$$

Let Φ be a Carathéodory operator and let S be the associated Schur operator. As in the previous section, U denotes the unitary operator $\mathscr{M}^\ell_{(1/\sqrt{2})(I+\Phi)} : \mathscr{H}_\ell(S) \to \mathscr{L}_\ell(\Phi)$. We will also consider the operator $V = \mathscr{M}^\ell_{(1/\sqrt{2})(I+\Phi_{[0]})} : \mathscr{D}_2 \to \mathscr{D}_2$. We claim that V is boundedly invertible. By [5, Theorem 7.3],

$$\Gamma_\Phi(\rho_W^{\wedge -1} E) = (\Phi + \Phi^\wedge(W)^*)\, \rho_W^{\wedge -1} E$$

and so

$$\|\Gamma_\Phi(\rho_W^{\wedge -1} E)\|^2_{\mathscr{L}_\ell(\Phi)} = \operatorname{Tr} E^* \rho_W^{\wedge -*} (\Phi^\wedge(W) + \Phi^\wedge(W)^*)\, \rho_W^{\wedge -1} E$$

is nonnegative for all $E \in \mathscr{D}_2$ and $W \in \Omega$. In particular, for $W = 0$ we have that

$$\operatorname{Tr} E^*(\Phi_{[0]} + \Phi_{[0]}^*)\, E \geqslant 0,$$

hence $\operatorname{Re} \Phi_{[0]} \geqslant 0$ and $(I + \Phi_{[0]})$ is boundedly invertible in $\ell^2_{\mathscr{N}}$. Now

$$\langle (I + \Phi_{[0]})\, E, (I + \Phi_{[0]})\, E \rangle_{\mathscr{D}_2} \geqslant \langle E, E \rangle_{\mathscr{D}_2}, \qquad E \in \mathscr{D}_2,$$

and the same is true also if we replace $\Phi_{[0]}$ by $\Phi_{[0]}^*$. By the closed graph theorem, the claim is true and we have

$$V^{-1} = \mathscr{M}^\ell_{\sqrt{2}(I+\Phi_{[0]})^{-1}}.$$

Since $S(I+\Phi)=(I-\Phi)$, we have that $S_{[0]}(I+\Phi_{[0]})=(I-\Phi_{[0]})$, and thus

$$S_{[0]}=(I+\Phi_{[0]})^{-1}(I-\Phi_{[0]}), \qquad \Phi_{[0]}=(I+S_{[0]})^{-1}(I-S_{[0]}).$$

LEMMA 4.2. *The formulas*

$$\begin{aligned} \mathbf{A}_\Phi(G) &= (G-G_{[0]})Z^{-1} \\ \mathbf{B}_\Phi(E) &= (\Phi-\Phi_{[0]})EZ^{-1} \\ \mathbf{C}_\Phi(G) &= G_{[0]} \\ \mathbf{D}_\Phi(E) &= \Phi_{[0]}E \end{aligned} \tag{4.3}$$

define a bounded colligation

$$\mathscr{V}_\Phi=\begin{pmatrix} \mathbf{A}_\Phi & \mathbf{B}_\Phi \\ \mathbf{C}_\Phi & \mathbf{D}_\Phi \end{pmatrix}:\begin{pmatrix} \mathscr{L}_\ell(\Phi) \\ \mathscr{D}_2 \end{pmatrix}\to\begin{pmatrix} \mathscr{L}_\ell(\Phi) \\ \mathscr{D}_2 \end{pmatrix}. \tag{4.4}$$

The operators $\mathbf{A}_\Phi$, $\mathbf{B}_\Phi$, $\mathbf{C}_\Phi$ and $\mathbf{D}_\Phi$ are related to the operators in the colligation (4.2) *of S by the equations*

$$\begin{aligned} \mathbf{A}_\Phi &= U\mathbf{A}_S U^* - \frac{1}{\sqrt{2}}U\mathbf{B}_S V\mathbf{C}_S U^* \\ \mathbf{B}_\Phi &= -U\mathbf{B}_S V \\ \mathbf{C}_\Phi &= V\mathbf{C}_S U^* \\ \mathbf{D}_\Phi &= I-\sqrt{2}\,V\mathbf{D}_S, \end{aligned} \tag{4.5}$$

and their adjoints are given by

$$\begin{aligned} \mathbf{A}_\Phi^*(\Gamma_\Phi(\rho_W^{\wedge\,-1}ZW^*EZ^*)) &= \Gamma_\Phi(\rho_W^{\wedge\,-1}E)-\Gamma_\Phi(\rho_0^{\wedge\,-1}E) \\ \mathbf{B}_\Phi^*(\Gamma_\Phi(\rho_W^{\wedge\,-1}ZW^*EZ^*)) &= (\Phi^\wedge(W)^*-\Phi_{[0]}^*)E \\ \mathbf{C}_\Phi^*(E) &= \Gamma_\Phi(\rho_0^{\wedge\,-1}E)=(\Phi+\Phi_{[0]}^*)E \\ \mathbf{D}_\Phi^*(E) &= \Phi_{[0]}^*E \end{aligned} \tag{4.6}$$

for all $E\in\mathscr{D}_2$ and $W\in\Omega$. The elements of the form $\Gamma_\Phi(\rho_W^{\wedge\,-1}ZW^*EZ^*)$ are total in $\mathscr{L}_\ell(\Phi)$.

Proof. In the following calculations $E, E_1, E_2\in\mathscr{D}_2$, $V, W\in\Omega$ and $G\in\mathscr{L}_\ell(\Phi)$ are arbitrary; then by Lemma 3.2, $F=\sqrt{2}(I+\Phi)^{-1}G\in\mathscr{H}_\ell(S)$. The formulas for $\mathbf{D}_\Phi$ and $\mathbf{C}_\Phi$ follow from

$$(I-\sqrt{2}\,V\mathbf{D}_S)(E) = E-(I+\Phi_{[0]})\,S_{[0]}E$$
$$= E-(I-\Phi_{[0]})\,E = \Phi_{[0]}E = \mathbf{D}_\Phi(E),$$
$$V\mathbf{C}_S U^*G = V\mathbf{C}_S(F) = \frac{1}{\sqrt{2}}\,(I+\Phi_{[0]})\,F_{[0]} = G_{[0]} = \mathbf{C}_\Phi(G).$$

We now turn to the equation for $\mathbf{B}_\Phi$:

$$-U\mathbf{B}_S VE = -\frac{1}{\sqrt{2}}\,(I+\Phi)(S-S_{[0]})\,\frac{1}{\sqrt{2}}\,(I+\Phi_{[0]})\,EZ^{-1}$$
$$= -\frac{1}{2}\,((I-\Phi)(I+\Phi_{[0]})-(I+\Phi)(I-\Phi_{[0]}))\,EZ^{-1}$$
$$= (\Phi-\Phi_{[0]})\,EZ^{-1}$$
$$= \mathbf{B}_\Phi(E).$$

Finally, we prove the formula for $\mathbf{A}_\Phi$:

$$\left(U\mathbf{A}_S U^* - \frac{1}{\sqrt{2}}\,U\mathbf{B}_S V\mathbf{C}_S U^*\right)(G)$$
$$= U\mathbf{A}_S(F) - \frac{1}{\sqrt{2}}\,U\mathbf{B}_S V\mathbf{C}_S(F)$$
$$= U((F-F_{[0]})\,Z^{-1}) - \frac{1}{\sqrt{2}}\,U\mathbf{B}_S V(F_{[0]})$$
$$= \frac{1}{\sqrt{2}}\,(I+\Phi)(F-F_{[0]})\,Z^{-1} - \frac{1}{\sqrt{2}}\,U\mathbf{B}_S\left(\frac{1}{\sqrt{2}}\,(I+\Phi_{[0]})\,F_{[0]}\right)$$
$$= \frac{1}{\sqrt{2}}\,(I+\Phi)(F-F_{[0]})\,Z^{-1} - \frac{1}{2}\,U((S-S_{[0]})(I+\Phi_{[0]})\,F_{[0]}Z^{-1})$$
$$= \frac{1}{\sqrt{2}}\,(I+\Phi)(F-F_{[0]})\,Z^{-1} - \frac{1}{2\sqrt{2}}\,(I+\Phi)(S-S_{[0]})(I+\Phi_{[0]})\,F_{[0]}Z^{-1}$$
$$= \frac{1}{\sqrt{2}}\,(I+\Phi)(F-F_{[0]})\,Z^{-1} - \frac{1}{2\sqrt{2}}\,((I-\Phi)(I+\Phi_{[0]})$$
$$-(I+\Phi)(I-\Phi_{[0]}))\,F_{[0]}Z^{-1}$$
$$= \left(\frac{1}{\sqrt{2}}\,(I+\Phi)\,F - \frac{1}{\sqrt{2}}\,(I+\Phi_{[0]})\,F_{[0]}\right)Z^{-1}$$
$$= \mathbf{A}_\Phi(G).$$

The formula for $\mathbf{D}_\Phi^*$ follows from

$$\begin{aligned}\langle \mathbf{D}_\Phi^*(E_1), E_2\rangle_{\mathscr{D}_2} &= \langle E_1, \mathbf{D}_\Phi(E_2)\rangle_{\mathscr{D}_2} = \langle E_1, \Phi_{[0]} E_2\rangle_{\mathscr{D}_2}\\ &= \mathrm{Tr}\, E_2^* \Phi_{[0]}^* E_1 = \langle \Phi_{[0]}^* E_1, E_2\rangle_{\mathscr{D}_2}.\end{aligned}$$

To prove the formula for $\mathbf{C}_\Phi^*$ we use (2.3) and (2.2):

$$\begin{aligned}\langle G, \mathbf{C}_\Phi^*(E)\rangle_{\mathscr{L}_\ell(\Phi)} &= \langle G_{[0]}, E\rangle_{\mathscr{D}_2} = \mathrm{Tr}\, G_{[0]}^* E = \langle G, \Gamma_\Phi(\rho_0^{\wedge\,-1} E)\rangle_{\mathscr{L}_\ell(\Phi)}\\ &= \langle G, (\Phi + \Phi_{[0]}^*)\, E\rangle_{\mathscr{L}_\ell(\Phi)}.\end{aligned}$$

Next we compute the adjoint of $\mathbf{B}_\Phi$,

$$\begin{aligned}\langle E, \mathbf{B}_\Phi^*(\Gamma_\Phi(\rho_W^{\wedge\,-1} E_1))\rangle_{\mathscr{D}_2} &= \langle (\Phi - \Phi_{[0]})\, EZ^*, \Gamma_\Phi(\rho_W^{\wedge\,-1} E_1)\rangle_{\mathscr{L}_\ell(\Phi)}\\ &= \mathrm{Tr}\, E_1^*((\Phi - \Phi_{[0]})\, EZ^{-1})^\wedge\,(W)\\ &= \mathrm{Tr}\, E_1^*((\Phi - \Phi_{[0]})\, Z^{-1})^\wedge\,(W)\, ZEZ^*,\end{aligned}$$

where, in the last equality, we used that $(XD)^\wedge\,(W) = X^\wedge(W)\, D$ for $X \in \mathscr{U}$ and $D \in \mathscr{D}$. It follows that

$$(\mathbf{B}_\Phi^*(\Gamma_\Phi(\rho_W^{\wedge\,-1} E_1)))^* = Z^* E_1^*((\Phi - \Phi_{[0]})\, Z^{-1})^\wedge\,(W)\, Z.$$

Now we replace E_1 by $ZW^*E_2Z^*$ and obtain that the right hand side is equal to

$$\begin{aligned}&(\mathbf{B}_\Phi^*(\Gamma_\Phi(\rho_W^{\wedge\,-1} ZW^*E_2Z^*)))^*\\ &\quad = E_2^*\, WZ^*((\Phi - \Phi_{[0]})\, Z^{-1})^\wedge\,(W)\, Z\\ &\quad = E_2^*\, WZ^* \left(\sum_{n=1}^\infty Z^{n-1}(Z\Phi_{[n]} Z^*)\right)^\wedge (W)\, Z\\ &\quad = E_2^* \left(\sum_{n=1}^\infty WZ^* W^{[n-1]} Z\Phi_{[n]}\right)\\ &\quad = E_2^* \left(\sum_{n=1}^\infty W^{[n]} \Phi_{[n]}\right)\\ &\quad = E_2^*(\Phi - \Phi_{[0]})^\wedge\,(W).\end{aligned}$$

We now prove the formula for $\mathbf{A}_{\Phi}^{*}$:

$$\langle \Gamma_{\Phi}(\rho_{V}^{\wedge\,-1} E_2), \mathbf{A}_{\Phi}^{*}(\Gamma_{\Phi}(\rho_{W}^{\wedge\,-1} Z W^{*} E_1 Z^{*}))\rangle_{\mathscr{L}_{\ell}(\Phi)}$$
$$= \langle (\Gamma_{\Phi}(\rho_{V}^{\wedge\,-1} E_2) - (\Gamma_{\Phi}(\rho_{V}^{\wedge\,-1} E_2))_{[0]})\, Z^{*}, \Gamma_{\Phi}(\rho_{W}^{\wedge\,-1} Z W^{*} E_1 Z^{*})\rangle_{\mathscr{L}_{\ell}(\Phi)}$$
$$= \operatorname{Tr} Z E_1^{*} W Z^{*}((\Gamma_{\Phi}\rho_{V}^{\wedge\,-1} - (\Gamma_{\Phi}\rho_{V}^{\wedge\,-1})_{[0]})\, E_2 Z^{*})^{\wedge}\,(W)$$
$$= \operatorname{Tr} E_1^{*} W Z^{*}((\Gamma_{\Phi}\rho_{V}^{\wedge\,-1} - (\Gamma_{\Phi}\rho_{V}^{\wedge\,-1})_{[0]})\, Z^{*}(Z E_2 Z^{*}))^{\wedge}\,(W)\, Z$$
$$= \operatorname{Tr} E_1^{*}(W Z^{*}(\Gamma_{\Phi}\rho_{V}^{\wedge\,-1} - (\Gamma_{\Phi}\rho_{V}^{\wedge\,-1})_{[0]})\, Z^{*})^{\wedge}\,(W)(Z E_2 Z^{*})\, Z$$
$$= \operatorname{Tr} E_1^{*}((\Gamma_{\Phi}\rho_{V}^{\wedge\,-1})^{\wedge}\,(W) - (\Gamma_{\Phi}\rho_{V}^{\wedge\,-1})_{[0]})\, E_2$$
$$= \langle \Gamma_{\Phi}\rho_{V}^{\wedge\,-1} E_2 - \Gamma_{\Phi} E_2, \rho_{W}^{\wedge\,-1} E_1\rangle_{\mathscr{U}_2}$$
$$= \langle \Gamma_{\Phi}\rho_{V}^{\wedge\,-1} E_2 - \Gamma_{\Phi} E_2, \Gamma_{\Phi}\rho_{W}^{\wedge\,-1} E_1\rangle_{\mathscr{L}_{\ell}(\Phi)}.$$

To complete the proof, it remains to show that the set of all operators of the form

$$\Gamma_{\Phi}(\rho_{W}^{\wedge\,-1} Z W^{*} E Z^{*})$$

is total in $\mathscr{L}_{\ell}(\Phi)$. Indeed if $F \in \mathscr{L}_{\ell}(\Phi)$ is orthogonal to these elements, then

$$0 = \langle F, \Gamma_{\Phi}(\rho_{W}^{\wedge\,-1} Z W^{*} E Z^{*})\rangle_{\mathscr{L}_{\ell}(\Phi)} = \operatorname{Tr} Z E^{*} W Z^{*} F^{\wedge}\,(W)$$
$$= \operatorname{Tr} E^{*}(W Z^{*} F^{\wedge}\,(W)\, Z)$$
$$= \operatorname{Tr} E^{*}(F Z)^{\wedge}\,(W),$$

hence $(FZ)^{\wedge}\,(W) = 0$ for any $W \in \Omega$, so $FZ = 0$ and thus $F = 0$. ∎

Before stating the main result of this section we need one more preliminary lemma:

LEMMA 4.3. *For $W \in \mathscr{U}$ with $\|W\| < 1$, the operator $\mathscr{M}^{r}_{W}$ of right multiplication by W is a strict contraction from $\mathscr{L}_{\ell}(\Phi)$ to itself.*

Proof. For $F = \sum_{i=1}^{k} \Gamma_{\Phi}\rho_{V_i}^{\wedge\,-1} E_i$, $E_i \in \mathscr{D}_2$, $V_i \in \Omega$, we have $\|F\|^2_{\mathscr{L}_{\ell}(\Phi)} = \operatorname{Tr}[F, F]_{\mathscr{L}_{\ell}(\Phi)}$, where $[F, F]_{\mathscr{L}_{\ell}(\Phi)}$ is the form

$$[F, F]_{\mathscr{L}_{\ell}(\Phi)} = \sum_{i,\, j=1}^{k} E_j^{*}(\Gamma_{\Phi}\rho_{V_i}^{\wedge\,-1})^{\wedge}\,(V_j)\, E_i$$
$$= \sum_{n} \left(\sum_{j=1}^{k} E_j^{*} V_j^{[n]}\right)\left(\sum_{j=1}^{k} E_j^{*} V_j^{[n]}\right)^{*}.$$

These equalities imply that $X=[F,F]_{\mathscr{L}_\ell(\Phi)}$ is of trace class, diagonal, and positive. Thus $X^{1/2}\in\mathscr{D}_2$ and

$$\|FW\|^2_{\mathscr{L}_\ell(\Phi)}=\operatorname{Tr} W^*[F,F]_{\mathscr{L}_\ell(\Phi)}W=\operatorname{Tr} W^*X^{1/2}X^{1/2}W=\|X^{1/2}W\|^2_{\mathscr{D}_2}$$

$$\leqslant\|W\|^2\,\|X^{1/2}\|^2_{\mathscr{D}_2}=\|W\|^2\operatorname{Tr}X=\|W\|^2\,\|F\|^2_{\mathscr{L}_\ell(\Phi)}.$$

This implies that the operator $\mathscr{M}^r_W$ is a strict contraction. ▮

We now state the the Herglotz representation of Φ with coisometric main operator.

THEOREM 4.4. *Let $\Phi\in\mathscr{U}(\ell^2_{\mathscr{N}})$ be a Carathéodory operator. For any $W\in\mathscr{D}(\ell^2_{\mathscr{B}})$ such that $\|W\|<1$ and any $E\in\mathscr{D}_2(\ell^2_{\mathscr{B}};\ell^2_{\mathscr{N}})$ we have that*

$$(\Phi E)^{\triangle}(W)=(\mathbf{D}_\Phi+\mathbf{C}_\Phi\mathscr{M}^r_W(I_{\mathscr{L}_\ell(\Phi)}-\mathbf{A}_\Phi\mathscr{M}^r_W)^{-1}\,\mathbf{B}_\Phi)(E)$$

or equivalently,

$$(\Phi E)^{\triangle}(W)=(i\operatorname{Im}\mathbf{D}_\Phi+\tfrac{1}{2}\mathbf{C}_\Phi(I_{\mathscr{L}_\ell(\Phi)}+\mathscr{M}^r_W\mathbf{A}_\Phi)(I_{\mathscr{L}_\ell(\Phi)}-\mathscr{M}^r_W\mathbf{A}_\Phi)^{-1}\,\mathbf{C}^*_\Phi)(E),$$

where the operators $\mathbf{A}_\Phi$, $\mathbf{B}_\Phi$, $\mathbf{C}_\Phi$ and $\mathbf{D}_\Phi$ are defined by (4.3). *The main operator $\mathbf{A}_\Phi$ is coisometric*, $\operatorname{Re}\mathbf{D}_\Phi=\frac{1}{2}\mathbf{C}_\Phi\mathbf{C}^*_\Phi$, $\mathbf{B}_\Phi=\mathbf{A}_\Phi\mathbf{C}^*_\Phi$, *and the colligation $\mathscr{V}_\Phi$ is closely outer connected*:

$$\mathscr{L}_\ell(\Phi)=\overline{\operatorname{span}}\{\operatorname{Ran}(\mathbf{A}^*_\Phi)^n\,\mathbf{C}^*_\Phi\mid n\geqslant 0\}.$$

Proof. We first show that the main operator $\mathbf{A}_\Phi$ is a coisometry. Since the colligation $\mathscr{V}_S$ is coisometric and by the definition of $\mathbf{A}_\Phi$ in (4.5), we have

$$\mathbf{A}_\Phi\mathbf{A}^*_\Phi=\left(U\mathbf{A}_SU^*-\frac{1}{\sqrt{2}}\,U\mathbf{B}_SV\mathbf{C}_SU^*\right)\left(U\mathbf{A}^*_SU^*-\frac{1}{\sqrt{2}}\,U\mathbf{C}^*_SV^*\mathbf{B}^*_SU^*\right)$$

$$=U\mathbf{A}_S\mathbf{A}^*_SU^*-\frac{1}{\sqrt{2}}\,U\mathbf{B}_SV\mathbf{C}_S\mathbf{A}^*_SU^*$$

$$-\frac{1}{\sqrt{2}}\,U\mathbf{A}_S\mathbf{C}^*_SV^*\mathbf{B}^*_SU^*+\frac{1}{2}\,U\mathbf{B}_SV\mathbf{C}_S\mathbf{C}^*_SV^*\mathbf{B}^*_SU^*$$

$$= U(I_{\mathscr{H}_\ell(S)} - \mathbf{B}_S\mathbf{B}_S^*)\, U^* + \frac{1}{\sqrt{2}}\, U\mathbf{B}_S V\mathbf{D}_S\mathbf{B}_S^* U^*$$

$$+ \frac{1}{\sqrt{2}}\, U\mathbf{B}_S\mathbf{D}_S^* V^*\mathbf{B}_S^* U^* + \frac{1}{2}\, U\mathbf{B}_S V(I_{\mathscr{D}_2} - \mathbf{D}_S\mathbf{D}_S^*)\, V^*\mathbf{B}_S^* U^*$$

$$= I_{\mathscr{L}_\ell(\Phi)} - U\mathbf{B}_S \left\{ I_{\mathscr{D}_2} - \frac{1}{\sqrt{2}}\, V\mathbf{D}_S - \frac{1}{\sqrt{2}}\, \mathbf{D}_S^* V^* \right.$$

$$\left. - \frac{1}{2}\, V(I_{\mathscr{D}_2} - \mathbf{D}_S\mathbf{D}_S^*)\, V^* \right\} \mathbf{B}_S^* U^*.$$

The expression in brackets is equal to 0 because for every $E \in \mathscr{D}_2$ we have

$$\left(I_{\mathscr{D}_2} - \frac{1}{\sqrt{2}}\, V\mathbf{D}_S - \frac{1}{\sqrt{2}}\, \mathbf{D}_S^* V^* - \frac{1}{2}\, V(I_{\mathscr{D}_2} - \mathbf{D}_S\mathbf{D}_S^*)\, V^* \right)(E)$$

$$= E - \frac{1}{2}\,(I + \Phi_{[0]})\, S_{[0]} E - \frac{1}{2}\, S_{[0]}^* (I + \Phi_{[0]}^*)\, E$$

$$- \frac{1}{4}\,(I + \Phi_{[0]})(I - S_{[0]} S_{[0]}^*)(I + \Phi_{[0]}^*)\, E$$

$$= E - \frac{1}{2}\,(I - \Phi_{[0]})\, E - \frac{1}{2}\,(I - \Phi_{[0]}^*)\, E$$

$$- \frac{1}{4}\,((I + \Phi_{[0]})(I + \Phi_{[0]}^*) - (I - \Phi_{[0]})(I - \Phi_{[0]}^*))\, E$$

$$= E - \frac{1}{2}\,(I - \Phi_{[0]})\, E - \frac{1}{2}\,(I - \Phi_{[0]}^*)\, E - \frac{1}{2}\,(\Phi_{[0]} + \Phi_{[0]}^*)\, E$$

$$= 0.$$

Hence $\mathbf{A}_\Phi \mathbf{A}_\Phi^* = I_{\mathscr{L}_\ell(\Phi)}$.

For $F \in \mathscr{L}_\ell(\Phi)$, $\mathscr{M}^r{}_W \mathbf{A}_\Phi(F) = (F - F_{[0]})\, Z^* W = \sum_{n=1}^\infty Z^n F_{[n]} Z^* W$, and so

$$\mathbf{C}_\Phi \mathscr{M}^r{}_W \mathbf{A}_\Phi(F) = Z F_{[1]} Z^* W.$$

In the same manner we get that for $n \geqslant 1$,

$$\mathbf{C}_\Phi (\mathscr{M}^r{}_W \mathbf{A}_\Phi)^n\, (F) = Z^n F_{[n]} (Z^* W)^n = F_{\{n\}}\, W^{\{n\}}.$$

Therefore

$$F^\triangle(W) = \mathbf{C}_\Phi (I_{\mathscr{L}_\ell(\Phi)} - \mathscr{M}^r{}_W \mathbf{A}_\Phi)^{-1}\, (F), \tag{4.7}$$

where the inverse exists because $\mathbf{A}_\Phi$ is a coisometry and $\mathscr{M}^r_W$ is a strict contraction by Lemma 4.3. Using that for all $F \in \mathscr{U}$ and $W \in \Omega$,

$$(FZ)^\triangle (W) = (FW)^\triangle (W) = F^\triangle (W^{(-1)}) W, \tag{4.8}$$

we get that

$$(FZ)^\triangle (W) = \mathscr{M}^r_W \mathbf{C}_\Phi (I_{\mathscr{L}_\ell(\Phi)} - \mathscr{M}^r_{W^{(-1)}} \mathbf{A}_\Phi)^{-1} (F).$$

Setting $F = \mathbf{B}_\Phi(E)$, we conclude that for all $E \in \mathscr{D}_2$ and $W \in \mathscr{D}$ with $\|W\| < 1$,

$$((\Phi - \Phi_{[0]}) E)^\triangle (W) = \mathscr{M}^r_W \mathbf{C}_\Phi (I_{\mathscr{L}_\ell(\Phi)} - \mathscr{M}^r_{W^{(-1)}} \mathbf{A}_\Phi)^{-1} \mathbf{B}_\Phi(E).$$

Since $\mathbf{D}_\Phi(E) = \Phi_{[0]} E$, $\mathscr{M}^r_{W^{(-1)}} \mathbf{A}_\Phi = \mathbf{A}_\Phi \mathscr{M}^r_W$ and $\mathscr{M}^r_W \mathbf{C}_\Phi = \mathbf{C}_\Phi \mathscr{M}^r_W$ we get the first representation formula, namely

$$(\Phi E)^\triangle (W) = (\mathbf{D}_\Phi + \mathbf{C}_\Phi \mathscr{M}^r_W (I_{\mathscr{L}_\ell(\Phi)} - \mathbf{A}_\Phi \mathscr{M}^r_W)^{-1} \mathbf{B}_\Phi)(E). \tag{4.9}$$

From $\mathbf{C}_\Phi \mathbf{C}_\Phi^*(E) = \mathbf{C}_\Phi(\Gamma_\Phi(\rho_0^{\wedge\,-1} E)) = (\Phi_{[0]} + \Phi_{[0]}^*) E = (2 \operatorname{Re} \mathbf{D}_\Phi)(E)$ we obtain

$$\operatorname{Re} \mathbf{D}_\Phi = \tfrac{1}{2} \mathbf{C}_\Phi \mathbf{C}_\Phi^*, \tag{4.10}$$

and from

$$\begin{aligned}
\mathbf{A}_\Phi \mathbf{C}_\Phi^*(E) &= \mathbf{A}_\Phi(\Gamma_\Phi(\rho_0^{\wedge\,-1} E)) \\
&= ((\Phi + \Phi_{[0]}^*) - (\Phi_{[0]} + \Phi_{[0]}^*)) EZ^{-1} \\
&= (\Phi - \Phi_{[0]}) EZ^{-1} \\
&= \mathbf{B}_\Phi(E)
\end{aligned}$$

we get

$$\mathbf{A}_\Phi \mathbf{C}_\Phi^* = \mathbf{B}_\Phi. \tag{4.11}$$

Thus, using (4.10), and (4.11) we get

$$\begin{aligned}
(\Phi E)^\triangle (W) &= (\mathbf{D}_\Phi + \mathbf{C}_\Phi \mathscr{M}^r_W (I_{\mathscr{L}_\ell(\Phi)} - \mathbf{A}_\Phi \mathscr{M}^r_W)^{-1} \mathbf{B}_\Phi)(E) \\
&= (i \operatorname{Im} \mathbf{D}_\Phi + \operatorname{Re} \mathbf{D}_\Phi + \mathbf{C}_\Phi (I_{\mathscr{L}_\ell(\Phi)} - \mathscr{M}^r_W \mathbf{A}_\Phi)^{-1} \mathscr{M}^r_W \mathbf{B}_\Phi)(E) \\
&= (i \operatorname{Im} \mathbf{D}_\Phi + \tfrac{1}{2} \mathbf{C}_\Phi \mathbf{C}_\Phi^* + \mathbf{C}_\Phi (I_{\mathscr{L}_\ell(\Phi)} - \mathscr{M}^r_W \mathbf{A}_\Phi)^{-1} \mathscr{M}^r_W \mathbf{A}_\Phi \mathbf{C}_\Phi^*)(E) \\
&= (i \operatorname{Im} \mathbf{D}_\Phi + \mathbf{C}_\Phi \{ \tfrac{1}{2} (I_{\mathscr{L}_\ell(\Phi)} - \mathscr{M}^r_W \mathbf{A}_\Phi) \\
&\quad + \mathscr{M}^r_W \mathbf{A}_\Phi \} (I_{\mathscr{L}_\ell(\Phi)} - \mathscr{M}^r_W \mathbf{A}_\Phi)^{-1} \mathbf{C}_\Phi^*)(E) \\
&= (i \operatorname{Im} \mathbf{D}_\Phi + \tfrac{1}{2} \mathbf{C}_\Phi (I_{\mathscr{L}_\ell(\Phi)} + \mathscr{M}^r_W \mathbf{A}_\Phi)(I_{\mathscr{L}_\ell(\Phi)} - \mathscr{M}^r_W \mathbf{A}_\Phi)^{-1} \mathbf{C}_\Phi^*)(E).
\end{aligned}$$

Finally, we prove that $\mathscr{V}_\Phi$ is closely outer-connected. Assume that $F \in \mathscr{L}_\ell(\Phi)$ is orthogonal to all operators of the form $(I_{\mathscr{L}_\ell(\Phi)} - \bar{\mu}\mathbf{A}_\Phi^*)^{-1}\,\mathbf{C}_\Phi^*(E)$ with $E \in \mathscr{D}_2$ and $\mu \in \mathbb{D}$. Then

$$\begin{aligned} 0 &= \langle F, (I_{\mathscr{L}_\ell(\Phi)} - \bar{\mu}\mathbf{A}_\Phi^*)^{-1}\,\mathbf{C}_\Phi^*(E)\rangle_{\mathscr{L}_\ell(\Phi)} \\ &= \langle \mathbf{C}_\Phi(I_{\mathscr{L}_\ell(\Phi)} - \mu\mathbf{A}_\Phi)^{-1}\,(F), E\rangle_{\mathscr{D}_2} = \langle F^\triangle(\mu I), E\rangle_{\mathscr{D}_2} \end{aligned}$$

which implies that $F^\triangle(\mu I) = 0$ for all $\mu \in \mathbb{D}$ and thus $F = 0$. ∎

COROLLARY 4.5. *Let*

$$\theta_{\mathscr{V}_\Phi}(\lambda) = \mathbf{D}_\Phi + \lambda\mathbf{C}_\Phi(I_{\mathscr{L}_\ell(\Phi)} - \lambda\mathbf{A}_\Phi)^{-1}\,\mathbf{B}_\Phi$$

be the characteristic function of the colligation $\mathscr{V}_\Phi$ associated with the Carathéodory operator Φ. Then

$$\frac{\theta_{\mathscr{V}_\Phi}(\lambda) + \theta_{\mathscr{V}_\Phi}(\mu)^*}{1 - \lambda\bar{\mu}} = \mathbf{C}_\Phi(I_{\mathscr{L}_\ell(\Phi)} - \lambda\mathbf{A}_\Phi)^{-1}\,(I_{\mathscr{L}_\ell(\Phi)} - \bar{\mu}\mathbf{A}_\Phi^*)^{-1}\,\mathbf{C}_\Phi^* \tag{4.12}$$

and hence the kernel is nonnegative in $\mathbb{D}$.

More generally, let $\mathscr{F}_W \colon \mathscr{D}_2 \to \mathscr{D}_2$ be the operator defined by

$$\mathscr{F}_W(E) = (\Phi E)^\triangle\,(W) = (\mathbf{D}_\Phi + \mathbf{C}_\Phi \mathscr{M}^r_W(I_{\mathscr{L}_\ell(\Phi)} - \mathbf{A}_\Phi\mathscr{M}^r_W)^{-1}\,\mathbf{B}_\Phi)(E).$$

Then, for $W, V \in \mathscr{D}$ with $\|W\|, \|U\| < 1$,

$$\begin{aligned} \mathscr{F}_W + \mathscr{F}_U^* = \mathbf{C}_\Phi&(I_{\mathscr{L}_\ell(\Phi)} - \mathscr{M}^r_W\mathbf{A}_\Phi)^{-1} \\ &\times \{I_{\mathscr{L}_\ell(\Phi)} - \mathscr{M}^r_W\mathscr{M}^{r*}_U\}(I_{\mathscr{L}_\ell(\Phi)} - \mathbf{A}_\Phi^*\mathscr{M}^r_{U^*})^{-1}\,\mathbf{C}_\Phi^*. \end{aligned}$$

Since $\theta_{\mathscr{V}_\Phi}(\lambda) = \mathscr{F}_{\lambda I}$, for $W = \lambda I$ and $V = \mu I$, $\lambda, \mu \in \mathbb{D}$, we have

$$\mathscr{F}_{\lambda I} + \mathscr{F}_{\mu I}^* = (1 - \lambda\bar{\mu})\,\mathbf{C}_\Phi(I_{\mathscr{L}_\ell(\Phi)} - \lambda\mathbf{A}_\Phi)^{-1}\,(I_{\mathscr{L}_\ell(\Phi)} - \bar{\mu}\mathbf{A}_\Phi^*)^{-1}\,\mathbf{C}_\Phi^*$$

and (4.12) follows.

THEOREM 4.6. *Let*

$$\mathscr{V} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} : \begin{pmatrix} \mathscr{H} \\ \mathscr{D}_2 \end{pmatrix} \to \begin{pmatrix} \mathscr{H} \\ \mathscr{D}_2 \end{pmatrix}$$

be a colligation such that the state space $\mathscr{H}$ is a right $\mathscr{D}$-module, the main operator $\mathbf{A}$ is coisometric, $\operatorname{Re}\mathbf{D} = (1/2)\,\mathbf{C}\mathbf{C}^*$, $\mathbf{A}\mathbf{C}^* = \mathbf{B}$, and such that $\mathscr{V}$ is closely outer-connected, that is, the set of all operators of the form

$(I_{\mathcal{H}}-\bar{\mu}\mathbf{A}^*)^{-1}\,\mathbf{C}^*(E)$, $E\in\mathcal{D}_2$, $\mu\in\mathbb{D}$, is total in $\mathcal{H}$. Let $\Phi\in\mathcal{U}$ be a Carathéodory operator and assume that for all $E\in\mathcal{D}_2$ and $W\in\mathcal{D}$ with $\|W\|<1$,

$$\begin{aligned}(\Phi E)^{\triangle}(W)&=(\mathbf{D}+\mathbf{C}\mathcal{M}^r_W(I_{\mathcal{H}}-\mathbf{A}\mathcal{M}^r_W)^{-1}\,\mathbf{B})(E)\\&=(i\,\mathrm{Im}\,\mathbf{D}+\tfrac{1}{2}\mathbf{C}(I_{\mathcal{H}}+\mathcal{M}^r_W\mathbf{A})(I_{\mathcal{H}}-\mathcal{M}^r_W\mathbf{A})^{-1}\,\mathbf{C}^*)(E).\end{aligned}$$

Then there exists a unitary map τ: $\mathcal{L}_\ell(\Phi)\to\mathcal{H}$, such that

$$\begin{pmatrix}\mathbf{A}&\mathbf{B}\\\mathbf{C}&\mathbf{D}\end{pmatrix}=\begin{pmatrix}\tau&0\\0&I\end{pmatrix}\begin{pmatrix}\mathbf{A}_\Phi&\mathbf{B}_\Phi\\\mathbf{C}_\Phi&\mathbf{D}_\Phi\end{pmatrix}\begin{pmatrix}\tau^*&0\\0&I\end{pmatrix}.$$

Proof. The hypotheses of the theorem imply that $\theta_{\mathcal{V}_\Phi}=\theta_{\mathcal{V}}$. By (4.12),

$$\frac{\theta_{\mathcal{V}_\Phi}(\lambda)+\theta_{\mathcal{V}_\Phi}(\mu)^*}{1-\lambda\bar{\mu}}=\mathbf{C}(I_{\mathcal{H}}-\lambda\mathbf{A})^{-1}\,(I_{\mathcal{H}}-\bar{\mu}\mathbf{A}^*)^{-1}\,\mathbf{C}^*. \tag{4.13}$$

Let $\mathbf{R}\subset\mathcal{L}_\ell(\Phi)\times\mathcal{H}$ be the linear relation spanned by all couples of the form

$$((I_{\mathcal{L}_\ell(\Phi)}-\bar{\mu}\mathbf{A}_\Phi^*)^{-1}\,\mathbf{C}_\Phi^*(E),\,(I_{\mathcal{H}}-\bar{\mu}\mathbf{A}^*)^{-1}\,\mathbf{C}^*(E)),\qquad E\in\mathcal{D}_2,\,\mu\in\mathbb{D}.$$

Then Dom $\mathbf{R}\subset\mathcal{L}_\ell(\Phi)$ and Ran $\mathbf{R}\subset\mathcal{H}$ are dense and, by (4.12) and (4.13),

$$\begin{aligned}&\langle(I_{\mathcal{L}_\ell(\Phi)}-\bar{\mu}\mathbf{A}_\Phi^*)^{-1}\,\mathbf{C}_\Phi^*(E),\,(I_{\mathcal{L}_\ell(\Phi)}-\bar{\lambda}\mathbf{A}_\Phi^*)^{-1}\,\mathbf{C}_\Phi^*(G)\rangle_{\mathcal{L}_\ell(\Phi)}\\&\quad=\langle\mathbf{C}_\Phi(I_{\mathcal{L}_\ell(\Phi)}-\lambda\mathbf{A}_\Phi)^{-1}\,(I_{\mathcal{L}_\ell(\Phi)}-\bar{\mu}\mathbf{A}_\Phi^*)^{-1}\,\mathbf{C}_\Phi^*(E),\,G\rangle_{\mathcal{D}_2}\\&\quad=\left\langle\frac{\theta_{\mathcal{V}_\Phi}(\lambda)+\theta_{\mathcal{V}_\Phi}(\mu)^*}{1-\lambda\bar{\mu}}(E),\,G\right\rangle_{\mathcal{D}_2}\\&\quad=\langle\mathbf{C}(I_{\mathcal{H}}-\lambda\mathbf{A})^{-1}\,(I_{\mathcal{H}}-\bar{\mu}\mathbf{A}^*)^{-1}\,\mathbf{C}^*(E),\,G\rangle_{\mathcal{D}_2}\\&\quad=\langle(I_{\mathcal{H}}-\bar{\mu}\mathbf{A}^*)^{-1}\,\mathbf{C}^*(E),\,(I_{\mathcal{H}}-\bar{\lambda}\mathbf{A}^*)^{-1}\,\mathbf{C}^*(G)\rangle_{\mathcal{H}},\end{aligned}$$

which shows that $\mathbf{R}$ is also isometric. Hence the closure of $\mathbf{R}$ is the graph of a unitary map τ: $\mathcal{L}_\ell(\Phi)\to\mathcal{H}$ with the property that for all $E\in\mathcal{D}_2$ and $\mu\in\mathbb{D}$,

$$\tau((I_{\mathcal{L}_\ell(\Phi)}-\bar{\mu}\mathbf{A}_\Phi^*)^{-1}\,\mathbf{C}_\Phi^*(E))=(I_{\mathcal{H}}-\bar{\mu}\mathbf{A}^*)^{-1}\,\mathbf{C}^*(E). \tag{4.14}$$

From the two formulas for $(\Phi E)^{\triangle}(W)$ with $W=0$ we get

$$\mathbf{D}=\mathbf{D}_\Phi. \tag{4.15}$$

By (4.14) with $\mu=0$, $\tau \mathbf{C}_{\Phi}^{*}=\mathbf{C}^{*}$ and therefore

$$\mathbf{C}=\mathbf{C}_{\Phi}\tau^{*}. \tag{4.16}$$

Also from the definition of τ we have that

$$\begin{aligned}\tau(\mathbf{C}_{\Phi}^{*}(E)&+\bar{\mu}\mathbf{A}_{\Phi}^{*}(I_{\mathscr{L}_{\ell}(\Phi)}-\bar{\mu}\mathbf{A}_{\Phi}^{*})^{-1}\,\mathbf{C}_{\Phi}^{*}(E))\\&=\mathbf{C}^{*}(E)+\bar{\mu}\mathbf{A}^{*}(I_{\mathscr{H}}-\bar{\mu}\mathbf{A}^{*})^{-1}\,\mathbf{C}^{*}(E)\end{aligned}$$

and using (4.16) we get

$$\begin{aligned}&\tau(\mathbf{A}_{\Phi}^{*}(I_{\mathscr{L}_{\ell}(\Phi)}-\bar{\mu}\mathbf{A}_{\Phi}^{*})^{-1}\,\mathbf{C}_{\Phi}^{*}(E))\\&\quad=\mathbf{A}^{*}(I_{\mathscr{H}}-\bar{\mu}\mathbf{A}^{*})^{-1}\,\mathbf{C}^{*}(E)=\mathbf{A}^{*}\tau(I_{\mathscr{L}_{\ell}(\Phi)}-\bar{\mu}\mathbf{A}_{\Phi}^{*})^{-1}\,\mathbf{C}_{\Phi}^{*}(E).\end{aligned}$$

This implies that $\tau\mathbf{A}_{\Phi}^{*}=\mathbf{A}^{*}\tau$ on a dense set and so, by the boundedness of $\mathbf{A}_{\Phi}$ and $\mathbf{A}$,

$$\mathbf{A}=\tau\mathbf{A}_{\Phi}\tau^{*}. \tag{4.17}$$

Finally, $\mathbf{B}_{\Phi}=\mathbf{A}_{\Phi}\mathbf{C}_{\Phi}^{*}$ and $\mathbf{B}=\mathbf{A}\mathbf{C}^{*}$ imply $\mathbf{B}_{\Phi}=\mathbf{A}_{\Phi}\mathbf{C}_{\Phi}^{*}=\tau^{*}\mathbf{A}\tau\tau^{*}\mathbf{C}^{*}=\tau^{*}\mathbf{A}\mathbf{C}^{*}=\tau^{*}\mathbf{B}$, and thus

$$\mathbf{B}=\tau\mathbf{B}_{\Phi}. \tag{4.18}$$

From (4.15)–(4.18) we conclude that the two colligations are unitarily equivalent. ∎

5. THE ISOMETRIC REPRESENTATION

For $F\in\mathscr{U}$ and $W\in\Omega$,

$$F^{\wedge}(W^{*})^{*}=\sum_{n=0}^{\infty}F_{[n]}^{*}Z^{*n}(ZW)^{n},\qquad F^{\triangle}(W^{*})^{*}=\sum_{n=0}^{\infty}(WZ)^{n}\,Z^{*n}F_{\{n\}}^{*}.$$

This gives rise to the analogs of the left and the right point evaluations for lower triangular operators: $G\in\mathscr{L}$ can formally be written as a right and a left power series in Z^{*}, namely $G=\sum_{n=0}^{\infty}G_{[n]}Z^{*n}$ and $G=\sum_{n=0}^{\infty}Z^{*n}G_{\{n\}}$ for some diagonal operators $G_{[n]}$ and $G_{\{n\}}$. We define the right point evaluation of G at $W\in\mathscr{X}$ by

$$G^{\vee}(W)=\sum_{n=0}^{\infty}G_{[n]}Z^{*n}(ZW)^{n}=\sum_{n=0}^{\infty}G_{[n]}\,W^{*[n]*}, \tag{5.1}$$

and the left point evaluation by

$$G^{\bigtriangledown}(W)=\sum_{n=0}^{\infty}(WZ)^n Z^{*n}G_{\{n\}}=\sum_{n=0}^{\infty} W^{*\{n\}*}G_{\{n\}}. \tag{5.2}$$

The four transforms are related by conjugation in the sense that for any $F\in\mathscr{U}$, the operator F^* belongs to $\mathscr{L}$ and hence for $W\in\Omega$

$$F^{\wedge}(W^*)^*=F^{*\vee}(W) \tag{5.3}$$

and

$$F^{\triangle}(W^*)^*=F^{*\bigtriangledown}(W). \tag{5.4}$$

When $\mathscr{N}=\mathbb{C}$, we note that $W^{*[n]*}=W^{[n]}$ and $W^{*\{n\}*}=W^{\{n\}}$.

The space $\mathscr{L}_2$ is a reproducing kernel Hilbert space with reproducing kernel

$$\sigma_W^{-1}=(I-W^*Z^*)^{-1}=\sum_{0}^{\infty} W^{\{n\}*}Z^{*n}$$

in the sense that for all $W\in\Omega$, $E\in\mathscr{D}_2$, and $G\in\mathscr{L}_2$, the operator $\sigma_W^{-1}E\in\mathscr{L}_2$ and

$$\langle G,\sigma_W^{-1}E\rangle_{\mathscr{L}_2}=\operatorname{Tr} E^*G^{\bigtriangledown}(W).$$

For $X\in\mathscr{L}$, denote by $\mathscr{M}_X^{\ell}$ the operator $G\mapsto XG$ from $\mathscr{L}_2$ to itself. It follows from the reproducing kernel formula that

$$\mathscr{M}_X^{\ell *}\sigma_W^{-1}E=(X^{\bigtriangledown}(W))^*\sigma_W^{-1}E. \tag{5.5}$$

Proposition 5.1. *Let $\Gamma\colon\mathscr{L}_2\to\mathscr{L}_2$ be a bounded nonnegative operator and let P be the orthogonal projection onto* $\operatorname{Ker}\Gamma$. *Then the operator range* $\operatorname{Ran}\Gamma^{1/2}$ *provided with the lifted norm*

$$\|\Gamma^{1/2}G\|_{\operatorname{Ran}\Gamma^{1/2}}=\|(I-P)\,G\|_{\mathscr{L}_2},\qquad G\in\mathscr{L}_2,$$

is a reproducing kernel Hilbert space with reproducing kernel $\Gamma\sigma_W^{-1}$ in the sense that for all $W\in\Omega$, $E\in\mathscr{D}_2$, and $G\in\operatorname{Ran}\Gamma^{1/2}$, the operator $\Gamma\sigma_W^{-1}E\in\operatorname{Ran}\Gamma^{1/2}$ and

$$\langle F,\Gamma\sigma_W^{-1}E\rangle_{\operatorname{Ran}\Gamma^{1/2}}=\operatorname{Tr} E^*F^{\bigtriangledown}(W). \tag{5.6}$$

Let $\Phi\in\mathscr{U}$ be a Carathéodory operator and let $S=(I+\Phi)^{-1}(I-\Phi)$ be the corresponding Schur operator in $\mathscr{U}$. Then $S^*=(I+\Phi^*)^{-1}(I-\Phi^*)$ is also a Schur operator and belongs to $\mathscr{L}$. The operator $\Gamma_{\Phi^*}=\mathscr{M}_{\Phi^*}^{\ell}+\mathscr{M}_{\Phi^*}^{\ell *}$

is a nonnegative mapping on $\mathscr{L}_2$. We denote by $\mathscr{L}_\ell(\Phi^*)$ the operator range $\operatorname{Ran} \Gamma_{\Phi^*}^{1/2}$ equipped with the lifted norm. By Proposition 5.1, it is a reproducing kernel with reproducing kernel $\Gamma_{\Phi^*}\sigma_W^{-1}$. An analog of Lemma 3.1 holds: We have

$$\mathscr{L}_\ell(\Phi^*) = \frac{1}{\sqrt{2}}\,(I+\Phi^*)\,\mathscr{H}_\ell(S^*),$$

and the multiplication operator

$$U = \mathscr{M}^\ell_{(1/\sqrt{2})(I+\Phi^*)} : \mathscr{H}_\ell(S^*) \to \mathscr{L}_\ell(\Phi^*)$$

is unitary. By V we will denote the multiplication operator

$$\mathscr{M}^\ell_{(1/\sqrt{2})(I+\Phi^*_{[0]})} : \mathscr{D}_2 \to \mathscr{D}_2$$

which is an invertible operator with inverse

$$V^{-1} = \mathscr{M}^\ell_{\sqrt{2}(I+\Phi^*_{[0]})^{-1}}.$$

In [7] it is shown that the colligation

$$\tilde{\mathscr{V}}_\ell = \begin{pmatrix} \tilde{\mathbf{A}}_\ell & \tilde{\mathbf{B}}_\ell \\ \tilde{\mathbf{C}}_\ell & \tilde{\mathbf{D}}_\ell \end{pmatrix} : \begin{pmatrix} \mathscr{H}_\ell(S^*) \\ \mathscr{D}_2 \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_\ell(S^*) \\ \mathscr{D}_2 \end{pmatrix} \tag{5.7}$$

defined by

$$\begin{aligned} \tilde{\mathbf{A}}_\ell(H) &= (H - H_{[0]})\,Z \\ \tilde{\mathbf{B}}_\ell(E) &= (S^* - S^*_{[0]})\,EZ \\ \tilde{\mathbf{C}}_\ell(H) &= H_{[0]} \\ \tilde{\mathbf{D}}_\ell(E) &= S^*_{[0]}\,E \end{aligned} \tag{5.8}$$

is bounded and coisometric. This result is the analog of Theorem 4.1 for lower triangular operators, and the starting point for obtaining the Herglotz realization for Φ with isometric main operator. Indeed, it can be shown that the colligation

$$\tilde{\mathscr{V}} = \begin{pmatrix} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & \tilde{\mathbf{D}} \end{pmatrix} : \begin{pmatrix} \mathscr{L}_\ell(\Phi^*) \\ \mathscr{D}_2 \end{pmatrix} \to \begin{pmatrix} \mathscr{L}_\ell(\Phi^*) \\ \mathscr{D}_2 \end{pmatrix} \tag{5.9}$$

defined by

$$\begin{aligned}\tilde{\mathbf{A}}(H) &= (H - H_{[0]})\, Z \\ \tilde{\mathbf{B}}(E) &= \Phi^* - \Phi^*_{[0]})\, EZ \\ \tilde{\mathbf{C}}(H) &= H_{[0]} \\ \tilde{\mathbf{D}}(E) &= \Phi^*_{[0]} E\end{aligned} \tag{5.10}$$

consists of continuous operators related to the operators in $\tilde{\mathscr{V}}_\ell$ via

$$\begin{aligned}\tilde{\mathbf{A}} &= U\tilde{\mathbf{A}}_\ell U^* - \frac{1}{\sqrt{2}}\, U\tilde{\mathbf{B}}_\ell V\tilde{\mathbf{C}}_\ell U^* \\ \tilde{\mathbf{B}} &= -U\tilde{\mathbf{B}}_\ell V \\ \tilde{\mathbf{C}} &= V\tilde{\mathbf{C}}_\ell U^* \\ \tilde{\mathbf{D}} &= I - \sqrt{2}\, V\tilde{\mathbf{D}}_\ell\end{aligned} \tag{5.11}$$

and its adjoint gives the following Herglotz representation for Φ with isometric main operator.

THEOREM 5.2. *Let $\Phi \in \mathscr{U}$ be a Carathéodory operator. For all $E \in \mathscr{D}_2$ and $W \in \mathscr{D}$ with $\|W\| < 1$,*

$$(\Phi E)^{\triangle}(W) = (\tilde{\mathbf{D}}^* + \tilde{\mathbf{B}}^*(I_{\mathscr{L}_\ell(\Phi^*)} - \tilde{\mathbf{A}}^* \mathscr{M}^r{}_{W^{(1)}})^{-1}\, \mathscr{M}^r{}_{W^{(1)}} \tilde{\mathbf{C}}^*)(E) \tag{5.12}$$

or equivalently,

$$\begin{aligned}(\Phi E)^{\triangle}(W) = (i\,\mathrm{Im}\,\tilde{\mathbf{D}}^* + \tfrac{1}{2}\tilde{\mathbf{C}}(I_{\mathscr{L}_\ell(\Phi^*)} + \tilde{\mathbf{A}}^*\mathscr{M}^r{}_{W^{(1)}}) \\ \times (I_{\mathscr{L}_\ell(\Phi^*)} - \tilde{\mathbf{A}}^*\mathscr{M}^r{}_{W^{(1)}})^{-1}\, \tilde{\mathbf{C}}^*)(E),\end{aligned} \tag{5.13}$$

where $\tilde{\mathbf{A}}$ is coisometric, $\mathrm{Re}\, \tilde{\mathbf{D}} = (1/2)\, \tilde{\mathbf{C}}\tilde{\mathbf{C}}^*$, $\tilde{\mathbf{A}}\tilde{\mathbf{C}}^* = \tilde{\mathbf{B}}$, *and the colligation $\tilde{\mathscr{V}}$ is closely outer connected. The latter properties determine $\tilde{\mathscr{V}}$ up to unitary equivalence.*

We only sketch the proof of the representations. Similar to the previous calculations for upper triangular operator we have that for $H = \sum_0^\infty H_{[n]} Z^{*n} \in \mathscr{L}_2$ and $W \in \mathscr{D}$ with $\|W\| < 1$,

$$\tilde{\mathbf{C}}(\mathscr{M}^r{}_W \tilde{\mathbf{A}})^n (H) = H_{[n]} Z^{*n} (ZW)^n = H_{[n]}\, W^{*[n]*},$$

so by (5.1)

$$H^{\vee}(W) = \tilde{\mathbf{C}}(I_{\mathscr{L}_\ell(\Phi^*)} - \mathscr{M}^r{}_W \tilde{\mathbf{A}})^{-1}(H),$$

and now one readily obtains

$$(\Phi^* E)^{\vee}\,(W) = (\tilde{\mathbf{D}} + \tilde{\mathbf{C}}\mathscr{M}^r_W(I_{\mathscr{L}_\ell(\Phi^*)} - \tilde{\mathbf{A}}\mathscr{M}^r_W)^{-1}\,\tilde{\mathbf{B}})(E) \tag{5.14}$$

or equivalently,

$$(\Phi^* E)^{\vee}\,(W) = (i\,\mathrm{Im}\,\tilde{\mathbf{D}} + \tfrac{1}{2}\tilde{\mathbf{C}}(I_{\mathscr{L}_\ell(\Phi^*)} + \mathscr{M}^r_W\tilde{\mathbf{A}})(I_{\mathscr{L}_\ell(\Phi^*)} - \mathscr{M}^r_W\tilde{\mathbf{A}})^{-1}\,\tilde{\mathbf{C}}^*)(E). \tag{5.15}$$

To obtain (5.13) from (5.15) we would like to take $E = I$ and take adjoints. But this method does not work because E is not a Hilbert–Schmidt operator. To avoid this method we use that $\mathscr{U}_2$ is also a reproducing kernel space with reproducing kernel $(I - W^*Z)^{-1}$ in the sense that for all $W \in \Omega$, $D \in \mathscr{D}_2$ and $F \in \mathscr{U}_2$, the operator $D(I - W^*Z)^{-1} \in \mathscr{U}_2$ and

$$\langle F, D(I - W^*Z)^{-1}\rangle_{\mathscr{U}_2} = \mathrm{Tr}\,D^*F^{\triangle}\,(W),$$

and we consider the operator $\mathscr{F}_W \colon \mathscr{D}_2 \to \mathscr{D}_2$ defined by

$$\mathscr{F}_W = i\,\mathrm{Im}\,\tilde{\mathbf{D}} + \tfrac{1}{2}\tilde{\mathbf{C}}(I_{\mathscr{L}_\ell(\Phi^*)} + \mathscr{M}^r_W\tilde{\mathbf{A}})(I_{\mathscr{L}_\ell(\Phi^*)} - \mathscr{M}^r_W\tilde{\mathbf{A}})^{-1}\,\tilde{\mathbf{C}}^*.$$

Then for any $E, D \in \mathscr{D}_2$,

$$\begin{aligned}
\langle \mathscr{F}^*_W(E), D\rangle_{\mathscr{D}_2} &= \langle E, (\Phi^* D)^{\vee}\,(W)\rangle_{\mathscr{D}_2}\\
&= \langle E, ((D^*\Phi)^{\wedge}\,(W^*))^*\rangle_{\mathscr{D}_2}\\
&= \mathrm{Tr}\,(D^*\Phi)^{\wedge}\,(W^*)\,E\\
&= \langle D^*\Phi, \rho_{\hat{W^*}}^{-1}\,E^*\rangle_{\mathscr{U}_2}\\
&= \mathrm{Tr}\,(E\rho_{\hat{W^*}}^{-*}\,D^*\Phi)\\
&= \langle \Phi E, D\rho_{\hat{W^*}}^{-1}\rangle_{\mathscr{U}_2}\\
&= \langle \Phi E, D(I - W^{(-1)*}Z)^{-1}\rangle_{\mathscr{U}_2}\\
&= \mathrm{Tr}\,D^*(\Phi E)^{\triangle}\,(W^{*(-1)})\\
&= \langle (\Phi E)^{\triangle}\,(W^{*(-1)}), D\rangle_{\mathscr{D}_2}.
\end{aligned}$$

Therefore $\mathscr{F}^*_W(E) = (\Phi E)^{\triangle}\,(W^{(-1)*})$ or equivalently, $\mathscr{F}^*_{W^{(1)*}}(E) = (\Phi E)^{\triangle}\,(W)$, and this implies (5.13).

6. THE UNITARY REPRESENTATION

In this section we derive a Herglotz representation for a Carathéodory operator Φ with unitary main operator acting in a state space $\mathscr{D}_\ell(\Phi)$ which we define below. We closely follow and use the analysis for Schur operators introduced in [7]. First we recall some results from [7]. If $S \in \mathscr{U}$ is a Schur operator, then the operator $(S^* - S^{*\vee}(W^*))(Z^* - W^*)^{-1} E$ belongs to $\mathscr{H}_\ell(S^*)$ for all $W \in \Omega$ and $E \in \mathscr{D}_2$. This fact allows to define a contraction operator from the space $\mathscr{H}_\ell(S)$ into the space $\mathscr{H}_\ell(S^*)$. More precisely, we have:

LEMMA 6.1. *The formula*

$$\mathbf{\Lambda}(\Gamma_S \rho_{\hat{W}}^{-1} E) = (S^* - S^{*\vee}(W^*))(Z^* - W^*)^{-1} E^{(1)}, \qquad W \in \Omega,\ E \in \mathscr{D}_2,$$

defines a contraction $\mathbf{\Lambda}\colon \mathscr{H}_\ell(S) \to \mathscr{H}_\ell(S^*)$. *The adjoint operator* $\mathbf{\Lambda}^*$ *from* $\mathscr{H}_\ell(S^*)$ *to* $\mathscr{H}_\ell(S)$ *is given by*

$$\mathbf{\Lambda}^*(\Gamma_{S^*} \sigma_W^{-1} E) = (S - S^{\triangle}(W^*))(Z - W^*)^{-1} E^{(-1)}, \qquad W \in \Omega,\ E \in \mathscr{D}_2.$$

The lemma implies that the operator

$$\Theta_S = \begin{pmatrix} I_{\mathscr{H}_\ell(S)} & \mathbf{\Lambda}^* \\ \mathbf{\Lambda} & I_{\mathscr{H}_\ell(S^*)} \end{pmatrix} : \begin{pmatrix} \mathscr{H}_\ell(S) \\ \mathscr{H}_\ell(S^*) \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_\ell(S) \\ \mathscr{H}_\ell(S^*) \end{pmatrix} \tag{6.1}$$

is nonnegative on the space $\mathscr{H}_\ell(S) \oplus \mathscr{H}_\ell(S^*)$. We denote by $\mathscr{D}_\ell(S)$ the operator range Ran $\Theta_S^{1/2}$ in $\mathscr{H}_\ell(S) \oplus \mathscr{H}_\ell(S^*)$ endowed with the lifted norm. We have seen that

$$U = \begin{pmatrix} \mathscr{M}^\ell_{(1/\sqrt{2})(I+\Phi)} & 0 \\ 0 & \mathscr{M}^\ell_{(1/\sqrt{2})(I+\Phi^*)} \end{pmatrix} : \begin{pmatrix} \mathscr{H}_\ell(S) \\ \mathscr{H}_\ell(S^*) \end{pmatrix} \to \begin{pmatrix} \mathscr{L}_\ell(\Phi) \\ \mathscr{L}_\ell(\Phi^*) \end{pmatrix}$$

is unitary (see Lemma 3.2). We define $\Theta_\Phi = U \Theta_S U^*$. It is a nonnegative map from $\mathscr{L}_\ell(\Phi) \oplus \mathscr{L}_\ell(\Phi^*)$ to itself. By $\mathscr{D}_\ell(\Phi)$ we denote the operator range Ran $\Theta_\Phi^{1/2}$ provided with the lifted norm. From $\Theta_\Phi^{1/2} = U \Theta_S^{1/2} U^*$ we see that as sets $\mathscr{D}_\ell(\Phi) = U \mathscr{D}_\ell(S)$. We claim that $U\colon \mathscr{D}_\ell(S) \to \mathscr{D}_\ell(\Phi)$ is unitary. To show this it suffices to show that (1) the set of elements of the form

$$\begin{pmatrix} F \\ G \end{pmatrix} = \Theta_S \begin{pmatrix} \Gamma_S \rho_{\hat{W}}^{-1} E_1 \\ \Gamma_{S^*} \sigma_W^{-1} E_2 \end{pmatrix}, \qquad W \in \Omega,\ E_1, E_2 \in \mathscr{D}_2,$$

is dense in $\mathscr{D}_\ell(S)$ and (2) that for these elements

$$\left\| U \begin{pmatrix} F \\ G \end{pmatrix} \right\|_{\mathscr{D}_\ell(\Phi)} = \left\| \begin{pmatrix} F \\ G \end{pmatrix} \right\|_{\mathscr{D}_\ell(S)}.$$

To see (1) assume that the element $\binom{F_1}{G_1} \in \mathscr{D}_\ell(S)$ is orthogonal to the set in (1). Then for all $W \in \Omega$ and $E_1, E_2 \in \mathscr{D}_2$,

$$\begin{aligned}
0 &= \left\langle \begin{pmatrix} F_1 \\ G_1 \end{pmatrix}, \Theta_S \begin{pmatrix} \Gamma_S \rho_{\hat{W}}^{-1} E_1 \\ \Gamma_{S^*} \sigma_W^{-1} E_2 \end{pmatrix} \right\rangle_{\mathscr{D}_\ell(S)} \\
&= \left\langle \begin{pmatrix} F_1 \\ G_1 \end{pmatrix}, \begin{pmatrix} \Gamma_S \rho_{\hat{W}}^{-1} E_1 \\ \Gamma_{S^*} \sigma_W^{-1} E_2 \end{pmatrix} \right\rangle_{\mathscr{H}_\ell(S) \oplus \mathscr{H}_\ell(S^*)} \\
&= \operatorname{Tr} E_1^* F_1^{\wedge}(W) + \operatorname{Tr} E_2^* G^{\nabla}(W),
\end{aligned}$$

which implies that $F_1 = 0$ and $G_1 = 0$. Part (2) follows from the equalities

$$\begin{aligned}
\left\| U \begin{pmatrix} F \\ G \end{pmatrix} \right\|^2_{\mathscr{D}_\ell(\Phi)} &= \left\| \Theta_\Phi U \begin{pmatrix} \Gamma_S \rho_{\hat{W}}^{-1} E_1 \\ \Gamma_{S^*} \sigma_W^{-1} E_2 \end{pmatrix} \right\|^2_{\mathscr{D}_\ell(\Phi)} \\
&= \left\langle \Theta_\Phi U \begin{pmatrix} \Gamma_S \rho_{\hat{W}}^{-1} E_1 \\ \Gamma_{S^*} \sigma_W^{-1} E_2 \end{pmatrix}, U \begin{pmatrix} \Gamma_S \rho_{\hat{W}}^{-1} E_1 \\ \Gamma_{S^*} \sigma_W^{-1} E_2 \end{pmatrix} \right\rangle_{\mathscr{L}_\ell(\Phi) \oplus \mathscr{L}_\ell(\Phi^*)} \\
&= \left\langle \Theta_S \begin{pmatrix} \Gamma_S \rho_{\hat{W}}^{-1} E_1 \\ \Gamma_{S^*} \sigma_W^{-1} E_2 \end{pmatrix}, \begin{pmatrix} \Gamma_S \rho_{\hat{W}}^{-1} E_1 \\ \Gamma_{S^*} \sigma_W^{-1} E_2 \end{pmatrix} \right\rangle_{\mathscr{H}_\ell(S) \oplus \mathscr{H}_\ell(S^*)} \\
&= \left\| \begin{pmatrix} F \\ G \end{pmatrix} \right\|^2_{\mathscr{D}_\ell(S)}.
\end{aligned}$$

The starting point for our main theorem, the Herglotz representation of Φ with unitary main operator, is the corresponding result for Schur operators; see [7, Theorems 7.1 and 7.2].

THEOREM 6.2. *The colligation*

$$\mathscr{W}_\ell = \begin{pmatrix} \alpha_\ell & \beta_{\Phi\ell} \\ \gamma_\ell & \delta_\ell \end{pmatrix} : \begin{pmatrix} \mathscr{D}_\ell(S) \\ \mathscr{D}_2 \end{pmatrix} \to \begin{pmatrix} \mathscr{D}_\ell(S) \\ \mathscr{D}_2 \end{pmatrix} \tag{6.2}$$

defined by

$$\begin{aligned}
\alpha_\ell \begin{pmatrix} F \\ H \end{pmatrix} &= \begin{pmatrix} (F - F_{[0]}) Z^{-1} \\ HZ^* - S^* F_{[0]} \end{pmatrix} \\
\beta_\ell(E) &= \begin{pmatrix} (S - S_{[0]}) EZ^{-1} \\ (I - S^* S_{[0]}) E \end{pmatrix} \\
\gamma_\ell \begin{pmatrix} F \\ H \end{pmatrix} &= F_{[0]}, \\
\delta_\ell(E) &= S_{[0]} E
\end{aligned} \tag{6.3}$$

is unitary and closely connected:

$$\mathscr{D}_\ell(S) = \overline{\operatorname{span}}\{\operatorname{Ran}(I_{\mathscr{D}_\ell(S)} - \lambda\alpha_\ell)^{-1}\beta_\ell, \operatorname{Ran}(I_{\mathscr{D}_\ell(S)} - \bar{\mu}\alpha_\ell^*)^{-1}\gamma_\ell^* \mid \lambda, \mu \in \mathbb{D}\}.$$

The adjoint colligation has entries given by

$$\begin{aligned}
\alpha_\ell^*\begin{pmatrix} F \\ H \end{pmatrix} &= \begin{pmatrix} FZ - SH_{[0]} \\ (H - H_{[0]})\, Z \end{pmatrix} \\
\beta_\ell^*\begin{pmatrix} F \\ H \end{pmatrix} &= H_{[0]} \\
\gamma_\ell^*(E) &= \begin{pmatrix} (I - SS_{[0]}^*)\, E \\ (S^* - S_{[0]}^*)\, EZ \end{pmatrix} \\
\delta_\ell^*(E) &= S_{[0]}^* E.
\end{aligned} \tag{6}$$

Besides the unitary operator U defined above we also use the multiplication operator $V = \mathscr{M}^\ell_{(1/\sqrt{2})(I + \Phi_{[0]})}: \mathscr{D}_2 \to \mathscr{D}_2$. We will use the following three lemmas.

LEMMA 6.3. *The colligation*

$$\mathscr{W}_\Phi = \begin{pmatrix} \alpha_\Phi & \beta_\Phi \\ \gamma_\Phi & \delta_\Phi \end{pmatrix} : \begin{pmatrix} \mathscr{D}_\ell(\Phi) \\ \mathscr{D}_2 \end{pmatrix} \to \begin{pmatrix} \mathscr{D}_\ell(\Phi) \\ \mathscr{D}_2 \end{pmatrix} \tag{6.5}$$

with

$$\begin{aligned}
\alpha_\Phi\begin{pmatrix} F \\ H \end{pmatrix} &= \begin{pmatrix} (F - F_{[0]})\, Z^{-1} \\ HZ^* - F_{[0]} \end{pmatrix} \\
\beta_\Phi(E) &= \begin{pmatrix} (\Phi - \Phi_{[0]})\, EZ^{-1} \\ -(\Phi^* + \Phi_{[0]})\, E \end{pmatrix} \\
\gamma_\Phi\begin{pmatrix} F \\ H \end{pmatrix} &= F_{[0]} \\
\delta_\Phi(E) &= \Phi_{[0]} E
\end{aligned} \tag{6.6}$$

is well defined, bounded and related to the colligation (6.3) *by the equations*

$$\alpha_{\Phi} = U\alpha_{\ell} U^* - \frac{1}{\sqrt{2}} U\beta_{\ell} V\gamma_{\ell} U^* \tag{6.7}$$

$$\beta_{\Phi} = -U\beta_{\ell} V \tag{6.8}$$

$$\gamma_{\Phi} = V\gamma_{\ell} U^* \tag{6.9}$$

$$\delta_{\Phi} = I - \sqrt{2}\, V\delta_{\ell}. \tag{6.10}$$

Proof. It suffices to prove the last four formulas. In the following $E \in \mathcal{D}_2$ is arbitrary. The fourth and the second equalities hold because

$$(I-\sqrt{2}\, V\delta_{\ell})(E) = E - (I+\Phi_{[0]})\, S_{[0]} E = E - (I-\Phi_{[0]})\, E = \Phi_{[0]} E = \delta_{\Phi}(E)$$

and

$$\begin{aligned}
U\beta_{\ell} V(E) &= U\beta_{\ell}\left(\frac{1}{\sqrt{2}}\,(I+\Phi_{[0]})\,E\right)\\
&= U\begin{pmatrix} (S-S_{[0]})\, 1/\sqrt{2}(I+\Phi_{[0]})\, EZ^{-1} \\ (I-S^*S_{[0]})\, 1/\sqrt{2}(I+\Phi_{[0]})\, E \end{pmatrix}\\
&= \begin{pmatrix} 1/\sqrt{2}(I+\Phi)(S-S_{[0]})\, 1/\sqrt{2}(I+\Phi_{[0]})\, EZ^{-1} \\ 1/\sqrt{2}(I+\Phi^*)(I-S^*S_{[0]})\, 1/\sqrt{2}(I+\Phi_{[0]})\, E \end{pmatrix}\\
&= \begin{pmatrix} 1/2((I-\Phi)(I+\Phi_{[0]}) - (I+\Phi)(I-\Phi_{[0]}))\, EZ^{-1} \\ 1/2((I+\Phi^*)(I+\Phi_{[0]}) - (I-\Phi^*)(I-\Phi_{[0]}))\, E \end{pmatrix}\\
&= \begin{pmatrix} -(\Phi-\Phi_{[0]})\, EZ^{-1} \\ (\Phi^*+\Phi_{[0]})\, E \end{pmatrix}\\
&= -\beta_{\Phi}(E).
\end{aligned}$$

To prove the other two relations, we write $\binom{F}{H} \in \mathcal{D}_{\ell}(\Phi)$ as $U\binom{F_1}{H_1}$ with $\binom{F_1}{H_1} \in \mathcal{D}_{\ell}(S)$. Then

$$\begin{aligned}
V\gamma_{\ell} U^* \begin{pmatrix} F \\ H \end{pmatrix} &= V\gamma_{\ell} \begin{pmatrix} F_1 \\ H_1 \end{pmatrix}\\
&= \frac{1}{\sqrt{2}}\,(I+\Phi_{[0]})(F_1)_{[0]} = \left(\frac{1}{\sqrt{2}}\,(I+\Phi)\, F_1\right)_{[0]} = \gamma_{\Phi}\begin{pmatrix} F \\ H \end{pmatrix},
\end{aligned}$$

which implies the third formula. Finally, the two summands in the fourth relation applied to $\binom{F}{H}$ become

$$\begin{aligned} U\alpha_\ell U^* \begin{pmatrix} F \\ G \end{pmatrix} &= U\alpha_\Phi \begin{pmatrix} F_1 \\ H_1 \end{pmatrix} \\ &= U \begin{pmatrix} (F_1 - (F_1)_{[0]})\, Z^{-1} \\ H_1 Z^* - S^*(F_1)_{[0]} \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{2}(I+\Phi)(F_1 - (F_1)_{[0]})\, Z^{-1} \\ 1/\sqrt{2}(I+\Phi^*)\, H_1 Z^* - 1/\sqrt{2}(I-\Phi^*)(F_1)_{[0]} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{\sqrt{2}}\, U\beta_\ell V\gamma_\ell U^* \begin{pmatrix} F \\ G \end{pmatrix} \\ &\quad = \frac{1}{\sqrt{2}}\, U\beta_\ell V\gamma_\ell \begin{pmatrix} F_1 \\ H_1 \end{pmatrix} \\ &\quad = \frac{1}{\sqrt{2}}\, U\beta_\ell V((F_1)_{[0]}) \\ &\quad = \frac{1}{\sqrt{2}}\, U\beta_\ell \left(\frac{1}{\sqrt{2}}\, (I+\Phi_{[0]})(F_1)_{[0]} \right) \\ &\quad = \frac{1}{\sqrt{2}}\, U \begin{pmatrix} (S - S_{[0]})\, 1/\sqrt{2}(I+\Phi_{[0]})(F_1)_{[0]}\, Z^{-1} \\ (I - S^* S_{[0]})\, 1/\sqrt{2}(I+\Phi_{[0]})(F_1)_{[0]} \end{pmatrix} \\ &\quad = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2}(I+\Phi)(S - S_{[0]})\, 1/\sqrt{2}(I+\Phi_{[0]})(F_1)_{[0]}\, Z^{-1} \\ 1/\sqrt{2}(I+\Phi^*)(I - S^* S_{[0]})\, 1/\sqrt{2}(I+\Phi_{[0]})(F_1)_{[0]} \end{pmatrix} \\ &\quad = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/2((I-\Phi)(I+\Phi_{[0]}) - (I+\Phi)(I-\Phi_{[0]}))(F_1)_{[0]}\, Z^{-1} \\ 1/2((I+\Phi^*)(I+\Phi_{[0]}) - (I-\Phi^*)(I-\Phi_{[0]}))(F_1)_{[0]} \end{pmatrix} \\ &\quad = \begin{pmatrix} -1/\sqrt{2}(\Phi - \Phi_{[0]})(F_1)_{[0]}\, Z^{-1} \\ 1/\sqrt{2}(\Phi^* + \Phi_{[0]})(F_1)_{[0]} \end{pmatrix}, \end{aligned}$$

and their difference is

$$\begin{pmatrix} (1/\sqrt{2}(I+\Phi)\, F_1 - 1/\sqrt{2}(I+\Phi_{[0]})(F_1)_{[0]})\, Z^{-1} \\ 1/\sqrt{2}(I+\Phi^*)\, H_1 Z^* - 1/\sqrt{2}(I+\Phi_{[0]})(F_1)_{[0]} \end{pmatrix} = \alpha_\Phi \begin{pmatrix} F \\ H \end{pmatrix}. \quad \blacksquare$$

LEMMA 6.4. *The entries of the adjoint colligation $\mathscr{W}_{\Phi}^{*}$ are given by*

$$\begin{aligned} \alpha_{\Phi}^{*}\begin{pmatrix} F \\ H \end{pmatrix} &= \begin{pmatrix} FZ - H_{[0]} \\ (H - H_{[0]})\, Z \end{pmatrix} \\ \beta_{\Phi}^{*}\begin{pmatrix} F \\ H \end{pmatrix} &= -H_{[0]} \\ \gamma_{\Phi}^{*}(E) &= \begin{pmatrix} (\Phi + \Phi_{[0]}^{*})\, E \\ -(\Phi^{*} - \Phi_{[0]}^{*})\, EZ \end{pmatrix} \\ \delta_{\Phi}^{*}(E) &= \Phi_{[0]}^{*} E. \end{aligned} \tag{6.11}$$

The computations are similar to the ones in the proof of the previous lemma and omitted.

LEMMA 6.5. *For $W \in \Omega$ with $\|W\| < 1$, the right multiplication operator $\mathscr{M}^{r}_{(W,\, W^{(1)})}$ defined by*

$$\mathscr{M}^{r}_{(W,\, W^{(1)})}\begin{pmatrix} F \\ H \end{pmatrix} = \begin{pmatrix} FW \\ HW^{(1)} \end{pmatrix}$$

is a strict contraction from $\mathscr{D}_{\ell}(\Phi)$ to itself.

Proof. We have $\mathscr{M}^{r}_{(W,\, W^{(1)})} U = U \mathscr{M}^{r}_{(W,\, W^{(1)})}$, where the multiplication operator on the right hand side is the right multiplication operator from $\mathscr{D}_{\ell}(S)$ to itself, which according to [7, Theorem 6.6], is a strict contraction. The lemma now follows since U is unitary. ▮

In the next theorem we introduce the unitary representation of Φ.

THEOREM 6.6. *Let $\Phi \in \mathscr{U}$ be a Carathéodory operator. For all $E \in \mathscr{D}_2$ and $W \in \mathscr{D}$ with $\|W\| < 1$,*

$$(\Phi E)^{\triangle}(W) = (\delta_{\Phi} + \gamma_{\Phi} \mathscr{M}^{r}_{(W,\, W^{(1)})}(I_{\mathscr{D}_{\ell}(\Phi)} - \alpha_{\Phi} \mathscr{M}^{r}_{(W,\, W^{(1)})})^{-1} \beta_{\Phi}(E) \tag{6.12}$$

or equivalently,

$$\begin{aligned} (\Phi E)^{\triangle}(W) = (i \operatorname{Im} \delta_{\Phi} + \tfrac{1}{2}\gamma_{\Phi}(I_{\mathscr{D}_{\ell}(\Phi)} + \mathscr{M}^{r}_{(W,\, W^{(1)})}\alpha_{\Phi})(I_{\mathscr{D}_{\ell}(\Phi)} \\ - \mathscr{M}^{r}_{(W,\, W^{(1)})}\alpha_{\Phi})^{-1} \gamma_{\Phi}^{*})(E). \end{aligned} \tag{6.13}$$

Similarly, we have for Φ^{*},

$$\begin{aligned} (\Phi^{*} E)^{\vee}(W) = (\delta_{\Phi}^{*} + \beta_{\Phi}^{*} \mathscr{M}^{r}_{(W^{(-1)},\, W)}(I_{\mathscr{D}_{\ell}(\Phi)} \\ - \alpha_{\Phi}^{*} \mathscr{M}^{r}_{(W^{(-1)},\, W)})^{-1} \gamma_{\Phi}^{*})(E) \end{aligned} \tag{6.14}$$

or equivalently,

$$\begin{aligned}(\Phi^*E)^\vee(W) &= (i\,\mathrm{Im}\,\delta_\Phi^* + \tfrac{1}{2}\beta_\Phi^*(I_{\mathscr{D}_\ell(\Phi)} \\ &\quad + \mathscr{M}^r_{(W^{(-1)},\,W)}\alpha_\Phi^*)(I_{\mathscr{D}_\ell(\Phi)} - \mathscr{M}^r_{(W^{(-1)},\,W)}\alpha_\Phi^*)^{-1}\beta_\Phi)(E).\end{aligned}$$

The operators α_Φ^* β_Φ^* γ_Φ *and* δ_Φ *are given by* (6.6), *the main operator* α_Φ *is unitary*, $\mathrm{Re}\,\delta_\Phi = 1/2\gamma_\Phi - {}_\Phi^*$, $\alpha_\Phi - {}_\Phi^* = \beta_\Phi$ *and the colligation* $\mathscr{W}_\Phi$ *in* (6.5) *is closely connected*:

$$\mathscr{D}_\ell(\Phi) = \overline{\mathrm{span}}\{\mathrm{Ran}(I_{\mathscr{D}_\ell(\Phi)} - \lambda\alpha_\Phi)^{-1}\beta_\Phi^*\ \mathrm{Ran}(I_{\mathscr{D}_\ell(\Phi)} - \bar{\mu}\alpha_\Phi^*)^{-1}\gamma_\Phi^* \mid \lambda, \mu \in \mathbb{D}\}.$$

Proof. We first show that the main operator α_Φ is unitary. Indeed, (6.7) and the fact that the colligation (6.2) is unitary imply that

$$\begin{aligned}\alpha_\Phi\alpha_\Phi^* &= \left(U\alpha_\ell U^* - \frac{1}{\sqrt{2}}\,U\beta_\ell V\gamma_\ell U^*\right)\left(U\alpha_\ell^* U^* - \frac{1}{\sqrt{2}}\,U\gamma_\ell^* V^*\beta_\ell U^*\right) \\ &= U\left(\alpha_\ell\alpha_\ell^* - \frac{1}{\sqrt{2}}\,\beta_\ell V\gamma_\ell\alpha_\ell^* - \frac{1}{\sqrt{2}}\,\alpha_\ell\gamma_\ell^* V^*\beta_\ell^* + \frac{1}{2}\,\beta_\ell V\gamma_\ell\gamma_\ell^* V^*\beta_\ell^*\right)U^* \\ &= U\left(I_{\mathscr{D}_\ell(S)} - \beta_\ell\beta_\ell^* + \frac{1}{\sqrt{2}}\,\beta_\ell V\delta_\ell\beta_\ell^*\right. \\ &\quad \left. + \frac{1}{\sqrt{2}}\,\beta_\ell\delta_\ell^* V^*\beta_\ell^* + \frac{1}{2}\,\beta_\ell V(I_{\mathscr{D}_2} - \delta_\ell\delta_\ell^*)\,V^*\beta_\ell^*\right)U^* \\ &= U\left(I_{\mathscr{D}_\ell(S)} - \beta_\ell\left\{I_{\mathscr{D}_2} - \frac{1}{\sqrt{2}}\,V\delta_\ell\right.\right. \\ &\quad \left.\left. - \frac{1}{\sqrt{2}}\,\delta_\ell^* V^* - \frac{1}{2}\,V(I_{\mathscr{D}_2} - \delta_\ell\delta_\ell^*)\,V^*\right\}\beta_\ell^*\right)U^*.\end{aligned}$$

The expression in $\{\ \}$ vanishes and so $\alpha_\Phi\alpha_\Phi^* = I_{\mathscr{D}_\ell(\Phi)}$. The relation $\alpha_\Phi^*\alpha_\Phi = I_{\mathscr{D}_l(\Phi)}$ can be proved in the same way.

Straightforward computations show that for $\binom{F}{H} \in \mathscr{D}_\ell(\Phi)$,

$$\gamma_\Phi(\mathscr{M}^r_{(W,\,W^{(1)})}\alpha_\Phi)^n\begin{pmatrix}F\\H\end{pmatrix} = Z^n F_{[n]}(Z^*W)^n, \qquad n \geqslant 1,$$

therefore

$$\gamma_\Phi(I_{\mathscr{D}_\ell(\Phi)} - \mathscr{M}^r_{(W,\,W^{(1)})}\alpha_\Phi)^{-1}\begin{pmatrix}F\\H\end{pmatrix} = F^\triangle(W). \tag{6.16}$$

Since α_Φ is unitary, the inverse exists by Lemma 6.5. On account of (4.8),

$$\mathscr{M}^r{}_W \gamma_\Phi (I_{\mathscr{D}_\ell(\Phi)} - \mathscr{M}^r{}_{(W^{(-1)}, W)} \alpha_\Phi)^{-1} \begin{pmatrix} F \\ H \end{pmatrix} = (FZ)^\triangle (W),$$

and hence with $\binom{F}{H} = \beta_\Phi(E)$, $E \in \mathscr{D}_2$,

$$(\mathscr{M}^r{}_W \gamma_\Phi (I_{\mathscr{D}_\ell(\Phi)} - \mathscr{M}^r{}_{(W^{(-1)}, W)} \alpha_\Phi)^{-1} \beta_\Phi)(E) = ((\Phi - \Phi_{[0]}) E)^\triangle (W).$$

Since $\delta_\Phi(E) = \Phi_{[0]} E$, $\mathscr{M}^r{}_{(W^{(-1)}, W)} \alpha_\Phi = \alpha^*_\Phi M^r_{(W, W^{(-1)})}$ and $\mathscr{M}^r{}_W \gamma_\Phi = \mathscr{M}^r{}_{(W, W^{(-1)})} \gamma_\Phi$ we get the representation formula (6.12) for Φ. For all $E \in \mathscr{D}_2$ we have

$$\gamma_\Phi \gamma^*_\Phi(E) = \gamma_\Phi \begin{pmatrix} (\Phi + \Phi^*_{[0]}) E \\ -(\Phi^* - \Phi^*_{[0]}) EZ \end{pmatrix} = (\Phi_{[0]} + \Phi^*_{[0]}) E$$

and

$$\begin{aligned} \alpha_\Phi \gamma^*_\Phi(E) &= \alpha_\Phi \begin{pmatrix} (\Phi + \Phi^*_{[0]}) E \\ -(\Phi^* - \Phi^*_{[0]}) EZ \end{pmatrix} \\ &= ((\Phi - \Phi_{[0]}) EZ^1 - (\Phi^* + \Phi_{[0]}) E) = \beta_\Phi(E), \end{aligned}$$

and thus $\operatorname{Re} \delta_\Phi = \frac{1}{2} \gamma_\Phi \gamma^*_\Phi$ and $\alpha_\Phi \gamma^*_\Phi = \beta^*_\Phi$ Using the representation formula (6.12) and these relations we obtain (6.13).

For $\binom{F}{H} \in \mathscr{D}_\ell(\Phi)$,

$$\beta^*_\Phi (\mathscr{M}^r{}_{(W^{(-1)}, W)} \alpha^*_\Phi)^n \begin{pmatrix} F \\ H \end{pmatrix} = -H_{[n]} Z^{*n} (ZW)^n, \qquad n \geqslant 1,$$

and hence

$$\beta^*_\Phi (I_{\mathscr{D}_\ell(\Phi)} - \mathscr{M}^r{}_{(W^{(-1)}, W)} \alpha^*_\Phi)^{-1} \begin{pmatrix} F \\ H \end{pmatrix} = -H^\vee (W). \tag{6.17}$$

Now we use that

$$(HZ^*)^\vee (W) = (HW)^\vee (W) = H^\vee (W^{(1)}) W, \qquad H \in \mathscr{L}, W \in \Omega,$$

and obtain

$$\mathscr{M}^r{}_W \beta^*_\Phi (I_{\mathscr{D}_\ell(\Phi)} - \mathscr{M}^r{}_{(W, W^{(1)})} \alpha^*_\Phi)^{-1} \begin{pmatrix} F \\ H \end{pmatrix} = -(HZ^*)^\vee (W),$$

which with $\binom{F}{H} = \gamma^*_\Phi(E)$ proves that

$$((\Phi^* - \Phi^*_{[0]}) E)^\vee (W) = \mathscr{M}^r{}_W \beta^*_\Phi (I_{\mathscr{D}_\ell(\Phi)} - \mathscr{M}^r{}_{(W, W^{(1)})} \alpha^*_\Phi)^{-1} \gamma^*_\Phi(E).$$

Since $\delta_\Phi^*(E) = \Phi_{[0]}^* E$, $\mathscr{M}^r_{(W, W^{(1)})}\alpha_\Phi^* = \alpha_\Phi^* \mathscr{M}^r_{(W^{(-1)}, W)}$ and $\mathscr{M}^r_W \beta_\Phi^* = \beta_\Phi^* \mathscr{M}^r_{(W^{(-1)}, W)}$ we get the formula (6.14) for Φ^*; (6.15) follows from (6.14) and $\operatorname{Re} \delta_\Phi = \frac{1}{2}\gamma_\Phi \gamma_\Phi^*$, $\alpha_\Phi \gamma_\Phi^* = \beta_\Phi$

The closely connectedness of the colligation $\mathscr{W}_\Phi$ follows from (6.16) and (6.17). Indeed, if the element $\binom{F}{H} \in \mathscr{D}_\ell(\Phi)$ is orthogonal to the span, then for all $E, D \in \mathscr{D}_2$ and $\mu \in \mathbb{D}$,

$$\begin{aligned}
0 &= \left\langle \begin{pmatrix} F \\ H \end{pmatrix}, ((I_{\mathscr{D}_\ell(\Phi)} - \bar{\mu}\alpha_\Phi^*)^{-1}\, \gamma_\Phi^* (I_{\mathscr{D}_\ell(\Phi)} - \bar{\mu}\alpha_\Phi)^{-1}\, \beta_\Phi) \begin{pmatrix} E \\ D \end{pmatrix} \right\rangle_{\mathscr{D}_\ell(\Phi)} \\
&= \left\langle \begin{pmatrix} \gamma_\Phi (I_{\mathscr{D}_\ell(\Phi)} - \mu\alpha_\Phi)^{-1} \\ \beta_\Phi^* (I_{\mathscr{D}_\ell(\Phi)} - \mu\alpha_\Phi^*)^{-1} \end{pmatrix} \begin{pmatrix} F \\ H \end{pmatrix}, \begin{pmatrix} E \\ D \end{pmatrix} \right\rangle_{\mathscr{D}_2 \oplus \mathscr{D}_2} \\
&= \left\langle \begin{pmatrix} F^\triangle(\mu I) \\ -H^\vee(\mu I) \end{pmatrix}, \begin{pmatrix} E \\ D \end{pmatrix} \right\rangle_{\mathscr{D}_2 \oplus \mathscr{D}_2},
\end{aligned}$$

hence $F^\triangle(\mu I) = 0$ and $H^\vee(\mu I) = 0$, $\mu \in \mathbb{D}$, which implies that $F = 0$ and $H = 0$. ∎

Corollary 6.7. *Let $\theta_{\mathscr{W}}$ be the characteristic function of the colligation $\mathscr{W} = \mathscr{W}_\Phi$ associated to the Carathéodory operator Φ,*

$$\theta_{\mathscr{W}}(\lambda) = i \operatorname{Im} \delta_\Phi + \tfrac{1}{2}\gamma_\Phi (I_{\mathscr{D}_L(\Phi)} + \lambda\alpha_\Phi)(I_{\mathscr{D}_L(\Phi)} - \lambda\alpha_\Phi)^{-1}\, \gamma_\Phi^*$$

and set $\widetilde{\theta_{\mathscr{W}}}(\lambda) = (\theta_{\mathscr{W}}(\bar{\lambda}))^*$. Then

$$\begin{pmatrix} \dfrac{\theta_{\mathscr{W}}(\lambda) + \theta_{\mathscr{W}}(\mu)^*}{1 - \lambda\bar{\mu}} & \dfrac{\theta_{\mathscr{W}}(\lambda) - \theta_{\mathscr{W}}(\bar{\mu})}{\lambda - \bar{\mu}} \\ \dfrac{\tilde{\theta}_{\mathscr{W}}(\lambda) - \tilde{\theta}_{\mathscr{W}}(\bar{\mu})}{\lambda - \bar{\mu}} & \dfrac{\tilde{\theta}_{\mathscr{W}}(\lambda) + \tilde{\theta}_{\mathscr{W}}(\mu)^*}{1 - \lambda\bar{\mu}} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_\Phi (I_{\mathscr{D}_\ell(\Phi)} - \lambda\alpha_\Phi)^{-1} \\ \beta_\Phi^* (I_{\mathscr{D}_\ell(\Phi)} - \lambda\alpha_\Phi^*)^{-1} \end{pmatrix} ((I_{\mathscr{D}_\ell(\Phi)} - \bar{\mu}\alpha_\Phi^*)^{-1}\, \gamma_\Phi^* (I_{\mathscr{D}_\ell(\Phi)} - \bar{\mu}\alpha_\Phi)^{-1}\, \beta_\Phi)$$

and hence the kernel is nonnegative in $\mathbb{D}$.

More generally, let $\mathscr{F}_W \colon \mathscr{D}_2 \to \mathscr{D}_2$ and $\mathscr{G}_W \colon \mathscr{D}_2 \to \mathscr{D}_2$ be the operators defined by

$$\mathscr{F}_W(E) = (\Phi E)^\triangle(W) = (\delta_\Phi + \gamma_\Phi \mathscr{M}^r_{(W, W^{(1)})} (I_{\mathscr{D}_\ell(\Phi)} - \alpha_\Phi \mathscr{M}^r_{(W, W^{(1)})})^{-1}\, \beta_\Phi)(E)$$

and

$$\begin{aligned}\mathscr{G}_W(E) &= (\Phi^* E)^{\vee}\,(W)\\ &= (\delta_\Phi^* + \beta_\Phi^* \mathscr{M}^r_{(W^{(-1)},\, W)}(I_{\mathscr{D}_\ell(\Phi)} - \alpha_\Phi^* \mathscr{M}^r_{(W^{(-1)},\, W)})^{-1}\,\gamma_\Phi^*)(E).\end{aligned}$$

Then for $W, V \in \mathscr{D}$, we have that $\mathscr{F}_W^* = \mathscr{G}_{W^{*(1)}}$ and

$$\begin{aligned}&\begin{pmatrix} \mathscr{F}_W + \mathscr{F}_V^* & \mathscr{F}_W - \mathscr{F}_{V^*} \\ \mathscr{G}_{W^{(1)}} - \mathscr{G}_{V^{*(1)}} & \mathscr{G}_{W^{(1)}} + \mathscr{G}_{V^{(1)}}^* \end{pmatrix}\\ &\quad= \begin{pmatrix} \gamma_\Phi (I_{\mathscr{D}_\ell(\Phi)} - \mathscr{M}^r_{(W,\, W^{(1)})}\alpha_\Phi)^{-1} & 0 \\ 0 & \beta_\Phi^* (I_{\mathscr{D}_\ell(\Phi)} - \mathscr{M}^r_{(W,\, W^{(1)})}\alpha_\Phi^*)^{-1} \end{pmatrix}\\ &\qquad\times \begin{pmatrix} I_{\mathscr{D}_\ell(\Phi)} - \mathscr{M}^r_{(W,\, W^{(1)})}\mathscr{M}^r_{(V^*,\, V^{*(1)})} & \mathscr{M}^r_{(W,\, W^{(1)})} - \mathscr{M}^r_{(V^*,\, V^{*(1)})} \\ \mathscr{M}^r_{(W,\, W^{(1)})} - \mathscr{M}^r_{(V^*,\, V^{*(1)})} & I_{\mathscr{D}_\ell(\Phi)} - \mathscr{M}^r_{(W,\, W^{(1)})}\mathscr{M}^r_{(V^*,\, ;\, V^{*(1)})} \end{pmatrix}\\ &\qquad\times \begin{pmatrix} (I_{\mathscr{D}_\ell(\Phi)} - \alpha_\Phi^* \mathscr{M}^r_{(V^*,\, V^{*(1)})})^{-1}\,\gamma_\Phi^* & 0 \\ 0 & (I_{\mathscr{D}_\ell(\Phi)} - \alpha_\Phi \mathscr{M}^r_{(V^*,\, V^{*(1)})})^{-1}\,\beta_\Phi \end{pmatrix}.\end{aligned}$$

The first equality follows from

$$\begin{aligned}\langle E, \mathscr{F}_W^*(G)\rangle_{\mathscr{D}_2} &= \langle (\Phi E)^{\triangle}\,(W), G\rangle_{\mathscr{D}_2}\\ &= \operatorname{Tr} G^*(\Phi E)^{\triangle}\,(W)\\ &= \operatorname{Tr} G^*\Phi E \rho_W^{\triangle -*}\\ &= \{\operatorname{Tr} E^*\Phi^* G \rho_W^{\triangle -1}\}^*\\ &= \{\operatorname{Tr} E^*\Phi^* G \rho_{W^{*(1)}}^{\vee\, -*}\}^*\\ &= \langle E, (\Phi^* G)^{\vee}\,(W^{*(1)})\rangle_{\mathscr{D}_2}.\end{aligned}$$

The proof of the second formula is straightforward. A similar formula appears in [13] in the setting of n-tuples of operators. Since $\theta_{\mathscr{W}}(\lambda) = \mathscr{F}_{\lambda I}$, this formula reduces to the formula in the corollary $W = \lambda I$, $V = \mu I$ where $\lambda, \mu \in \mathbb{D}$.

In the next theorem we study the uniqueness of the representations in Theorem 6.6.

THEOREM 6.8. *Let* $\widetilde{\mathscr{W}} = \begin{pmatrix} \tilde\alpha & \tilde\beta \\ \tilde\gamma & \tilde\delta \end{pmatrix}: \mathscr{H} \oplus \mathscr{D}_2 \to \mathscr{H} \oplus \mathscr{D}_2$ *be a closely connected colligation such that the state space* $\mathscr{H}$ *is a right* $\mathscr{D}$*-module, the main operator* $\tilde\alpha$ *is unitary,* $\operatorname{Re}\tilde\delta = \frac{1}{2}\tilde\gamma\tilde\gamma^*$, *and* $\tilde\alpha\tilde\gamma^* = \tilde\beta$. *Let* $\Phi \in \mathscr{U}$ *be a Carathéodory operator and assume that for all* $E \in \mathscr{D}_2$ *and* $W \in \mathscr{D}$ *with* $\|W\| < 1$,

$$(\Phi E)^{\triangle}\,(W) = (\tilde\delta + \tilde\gamma \mathscr{M}^r_W (I_{\mathscr{H}} - \tilde\alpha \mathscr{M}^r_W)^{-1}\,\tilde\beta)(E).$$

Then the colligations $\mathscr{W}_\Phi$ in (6.5) and $\widetilde{\mathscr{W}}$ are unitarily equivalent, that is, there exists a unitary map $\tau\colon \mathscr{D}_\ell(\Phi) \to \mathscr{H}$ such that

$$\begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix} = \begin{pmatrix} \tau & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \alpha_\Phi & \beta_\Phi \\ \gamma_\Phi & \delta_\Phi \end{pmatrix} \begin{pmatrix} \tau^* & 0 \\ 0 & I \end{pmatrix}.$$

Proof. We define a linear relation $\mathbf{R} \subset \mathscr{D}_\ell(\Phi) \times \mathscr{H}$ as the span of all couples of the form

$$\left(((I_{\mathscr{D}_\ell(\Phi)} - \bar{\mu}\alpha_\Phi^*)^{-1}\, \gamma_\Phi^*\, (I_{\mathscr{D}_\ell(\Phi)} - \bar{\mu}\alpha_\Phi)^{-1}\, \beta_\Phi) \begin{pmatrix} E \\ D \end{pmatrix}, \right.$$

$$\left. ((I_{\mathscr{H}} - \bar{\mu}\tilde{\alpha}^*)^{-1}\, \tilde{\gamma}^*\, (I_{\mathscr{H}} - \bar{\mu}\tilde{\alpha})^{-1}\, \tilde{\beta}) \begin{pmatrix} E \\ D \end{pmatrix} \right),$$

where $E, D \in \mathscr{D}_2$ and $\mu \in \mathbb{D}$. Then $\mathbf{R}$ has a dense range and a dense domain. Since $\theta_{\mathscr{W}}(\lambda) = \theta_{\widetilde{\mathscr{W}}}(\lambda)$ with $\mathscr{W} = \mathscr{W}_\Phi$ and by Corollary 6.7,

$$\left\| ((I_{\mathscr{D}_\ell(\Phi)} - \bar{\mu}\alpha_\Phi^*)^{-1}\, \gamma_\Phi^*\, (I_{\mathscr{D}_\ell(\Phi)} - \bar{\mu}\alpha_\Phi)^{-1}\, \beta_\Phi) \begin{pmatrix} E \\ D \end{pmatrix} \right\|^2_{\mathscr{D}_\ell(\Phi)}$$

$$= \left\langle \begin{pmatrix} \dfrac{\theta_{\mathscr{W}}(\lambda) + \theta_{\mathscr{W}}(\mu)^*}{1 - \lambda\bar{\mu}} & \dfrac{\theta_{\mathscr{W}}(\lambda) - \theta_{\mathscr{W}}(\bar{\mu})}{\lambda - \bar{\mu}} \\ \dfrac{\tilde{\theta}_{\mathscr{W}}(\lambda) - \tilde{\theta}_{\mathscr{W}}(\bar{\mu})}{\lambda - \bar{\mu}} & \dfrac{\tilde{\theta}_{\mathscr{W}}(\lambda) + \tilde{\theta}_{\mathscr{W}}(\mu)^*}{1 - \lambda\bar{\mu}} \end{pmatrix} \begin{pmatrix} E \\ D \end{pmatrix}, \begin{pmatrix} E \\ D \end{pmatrix} \right\rangle_{\mathscr{D}_2 \oplus \mathscr{D}_2}$$

$$= \left\| ((I_{\mathscr{H}} - \bar{\mu}\tilde{\alpha}^*)^{-1}\, \tilde{\gamma}^*\, (I_{\mathscr{H}} - \bar{\mu}\tilde{\alpha})^{-1}\, \tilde{\beta}) \begin{pmatrix} E \\ D \end{pmatrix} \right\|^2_{\mathscr{H}},$$

which implies that $\mathbf{R}$ is also isometric. Hence the closure of $\mathbf{R}$ is the graph of a unitary operator $\tau\colon \mathscr{D}_\ell(\Phi) \to \mathscr{H}$ with

$$\tau \left(((I_{\mathscr{D}_\ell(\Phi)} - \bar{\mu}\alpha_\Phi^*)^{-1}\, \gamma_\Phi^*\, (I_{\mathscr{D}_\ell(\Phi)} - \bar{\mu}\alpha_\Phi)^{-1}\, \beta_\Phi) \begin{pmatrix} E \\ D \end{pmatrix} \right)$$

$$= ((I_{\mathscr{H}} - \bar{\mu}\tilde{\alpha}^*)^{-1}\, \tilde{\gamma}^*\, (I_{\mathscr{H}} - \bar{\mu}\tilde{\alpha})^{-1}\, \tilde{\beta}) \begin{pmatrix} E \\ D \end{pmatrix}.$$

From the formulas for $(\Phi E)^\triangle(W)$ with $W = 0$ we have that

$$\tilde{\delta} = \delta_\Phi. \tag{6.18}$$

From the definition of τ with $\mu = 0$ we get

$$\tau\left((\gamma_{\Phi}^{*} \quad \beta_{\Phi}) \begin{pmatrix} E \\ G \end{pmatrix} \right) = (\tilde{\gamma}^{*} \quad \tilde{\beta}) \begin{pmatrix} E \\ G \end{pmatrix},$$

hence

$$\tilde{\beta} = \tau \beta_{\Phi}, \qquad \tilde{\gamma} = \gamma_{\Phi} \tau^{*}. \tag{6.19}$$

Also from the definition of τ, we have for all n, $m \geqslant 0$, $\tau \alpha_{\Phi}^{*n} \gamma_{\Phi}^{*} = \tilde{\alpha}^{*n} \tilde{\gamma}^{*}$, $\tau \alpha_{\Phi}^{m} \beta_{\Phi} = \tilde{\alpha}^{m} \tilde{\beta}$. Thus

$$\begin{aligned} &\tau \alpha_{\Phi}^{*}((I_{\mathscr{D}_{\ell}(\Phi)} - \bar{\mu} \alpha_{\Phi}^{*})^{-1} \, \gamma_{\Phi}^{*} \, (I_{\mathscr{D}_{\ell}(\Phi)} - \bar{\mu} \alpha_{\Phi})^{-1} \, \beta_{\Phi}) \begin{pmatrix} E \\ D \end{pmatrix} \\ &\quad = \tilde{\alpha}^{*}((I_{\mathscr{H}} - \bar{\mu} \tilde{\alpha}^{*})^{-1} \, \tilde{\gamma}^{*} \, (I_{\mathscr{H}} - \bar{\mu} \tilde{\alpha})^{-1} \, \tilde{\beta}) \begin{pmatrix} E \\ D \end{pmatrix} \\ &\quad = \tilde{\alpha}^{*} \tau((I_{\mathscr{D}_{\ell}(\Phi)} - \bar{\mu} \alpha_{\Phi}^{*})^{-1} \, \gamma_{\Phi}^{*} \, (I_{\mathscr{D}_{\ell}(\Phi)} - \bar{\mu} \alpha_{\Phi})^{-1} \, \beta_{\Phi}) \begin{pmatrix} E \\ D \end{pmatrix} \end{aligned}$$

and therefore $\tau \alpha_{\Phi}^{*} = \tilde{\alpha}^{*} \tau$ or equivalently,

$$\tilde{\alpha} = \tau \alpha_{\Phi} \tau^{*}. \tag{6.20}$$

From (6.18)–(6.20) we conclude that the colligations $\mathscr{W}_{\Phi}$ and $\tilde{\mathscr{W}}$ are unitarily equivalent. ∎

7. NONSTATIONARY ANALOG OF THE HERGLOTZ MEASURE

The classical Herglotz representation theorem of a Carathéodory function (see, for example, [26]) has been generalized to operator-valued functions in for example [18]:

THEOREM 7.1. *Let $T(\lambda)$ be an analytic function in $\mathbb{D}$ with values in $\mathscr{L}(\mathscr{H})$ where $\mathscr{H}$ is a separable Hilbert space. Then* $\operatorname{Re} T(\lambda) \geqslant 0$ *in $\mathbb{D}$ if and only if there exists a finite nonnegative Borel measure $d\mu$ on $\mathbb{T}$ with values in $\mathscr{L}(\mathscr{H})$ such that in the weak sense*

$$T(\lambda) \overset{w}{=} i \operatorname{Im} T(0) + \int_{0}^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda} \, d\mu(e^{it}), \qquad \lambda \in \mathbb{D}.$$

In this section we describe the nonstationary analog of the measure $d\mu$ using the unitary representation from Section 6. Let $\mathcal{J}\colon \ell^2_{\mathcal{N}} \to \mathcal{D}_2(\ell^2_{\mathbb{C}}; \ell^2_{\mathcal{N}})$ be the identification operator defined by:

$$v = \begin{pmatrix} \vdots \\ v_{-1} \\ \boxed{v_0} \\ v_1 \\ \vdots \end{pmatrix} \xmapsto{\mathcal{J}} V = \begin{pmatrix} \ddots & & & \\ & v_{-1} & & \\ & & \boxed{v_0} & \\ & & & v_1 \\ & & & & \ddots \end{pmatrix}. \tag{7.21}$$

Then $\mathcal{J}$ is a unitary operator with the properties:

$$\mathcal{J}(Z^n v) = Z^n \mathcal{J}(v)\, Z^{*n} = Z^n V Z^{*n} = V^{(-n)}$$

and for every $G \in \mathcal{D}(\ell^2_{\mathcal{N}})$,

$$\mathcal{J}(Gv) = G\mathcal{J}(v) = GV.$$

Lemma 7.2. *If $F \in \mathcal{U}$, then the series $F(\lambda) = \sum_{n=0}^{\infty} \lambda^n F_{\{n\}} Z^n$, $\lambda \in \mathbb{D}$, converges weakly to F as λ tends radially to* 1, *that is,*

$$F \overset{w}{=} \lim_{\lambda \to 1^-} F(\lambda). \tag{7.22}$$

Proof. Let $u \in \ell^2_{\mathcal{N}}$ and $v \in \ell^2_{\mathcal{M}}$. Using Lemma 2.1 and Abel's limit theorem we get

$$\begin{aligned} \langle Fu, v\rangle_{\ell^2_{\mathcal{M}}} &= \left\langle \sum_{n=0}^{\infty} F_{\{n\}} Z^n u, v \right\rangle_{\ell^2_{\mathcal{M}}} \\ &= \sum_{n=0}^{\infty} \langle F_{\{n\}} Z^n u, v\rangle_{\ell^2_{\mathcal{M}}} \\ &= \lim_{\lambda \to 1^-} \sum_{n=0}^{\infty} \lambda^n \langle F_{\{n\}} Z^n u, v\rangle_{\ell^2_{\mathcal{M}}} \\ &= \lim_{\lambda \to 1^-} \langle F(\lambda)\, u, v\rangle_{\ell^2_{\mathcal{M}}}. \quad \blacksquare \end{aligned}$$

Let $\Phi \in \mathscr{U}$ be a Carathéodory operator. By Theorem 6.6 with $W = \lambda I$, we have

$$
\begin{aligned}
(\Phi G)^{\triangle}(\lambda I) &= \sum_{n=0}^{\infty} \lambda^n \Phi_{\{n\}} Z^n G Z^{*n} \\
&= (i \operatorname{Im} \delta_{\Phi} + \tfrac{1}{2} \gamma_{\Phi\Phi} (I_{\mathscr{D}_L(\Phi)} + \lambda \alpha_{\Phi})(I_{\mathscr{D}_L(\Phi)} - \lambda \alpha_{\Phi})^{-1} \gamma_{\Phi}^{*})(G) \\
&= \theta_{\mathscr{W}_{\Phi}}(\lambda)(G), \qquad (7.23)
\end{aligned}
$$

where $\theta_{\mathscr{W}_{\Phi}}(\lambda)$ is the characterictic function of the colligation $\mathscr{W}_{\Phi}$; see Corollary 6.7.

LEMMA 7.3. *Let $\Phi \in \mathscr{U}$. If Φ is a Carathéodory operator, then the operator-valued function $\Phi(\lambda)$ has a nonnegative real part in $\mathbb{D}$ and*

$$
\Phi(\lambda) = \mathscr{J}^{*} \Theta_{\mathscr{V}_{\Phi}}(\lambda)\, \mathscr{J}. \qquad (7.24)
$$

Conversely, if $\Phi(\lambda)$ has a nonnegative real part, then Φ has a nonnegative real part.

Proof. Let $g \in \ell^2_{\mathscr{N}}$ and set $G = \mathscr{J}(g)$. Then using the properties of $\mathscr{J}$ and $\Theta_{\mathscr{V}_{\Phi}}(\lambda)$ we have

$$
\begin{aligned}
\Phi(\lambda)(g) &= \sum_{n=0}^{\infty} \lambda^n \Phi_{\{n\}} Z^n(g) \\
&= \sum_{n=0}^{\infty} \lambda^n \Phi_{\{n\}} Z^n \mathscr{J}^{*} \mathscr{J}(g) \\
&= \mathscr{J}^{*} \left(\sum_{n=0}^{\infty} \lambda^n \Phi_{\{n\}} Z^n G Z^{*n} \right) \\
&= \mathscr{J}^{*}((\Phi G)^{\triangle}(\lambda I)) \\
&= \mathscr{J}^{*} \Theta_{\mathscr{V}_{\Phi}}(\lambda)(G) \\
&= \mathscr{J}^{*} \Theta_{\mathscr{V}_{\Phi}}(\lambda)\, \mathscr{J}(g).
\end{aligned}
$$

Thus (7.24) holds. Using the fact that $\theta_{\mathscr{V}_{\Phi}}(\lambda)$ has a nonnegative real part in $\mathbb{D}$ (see Corollary 6.7) we conclude that $\Phi(\lambda)$ has a nonnegative real part in $\mathbb{D}$. For the converse, use Lemma 7.2 which implies that $\operatorname{Re} \Phi \overset{w}{=} \lim_{\lambda \to 1^-} \operatorname{Re} \Phi(\lambda)$. ∎

THEOREM 7.4. *Let $\Phi \in \mathscr{U}$ be a Carathéodory operator. Then there exists a finite nonnegative Borel measure $d\mu(e^{it})$ with values in $\mathscr{L}(\mathscr{D}_2)$ such that for $G \in \mathscr{D}_2$,*

$$\begin{aligned}\theta_{\mathscr{W}_\Phi}(\lambda)(G) &= (\Phi G)^{\triangle}\,(\lambda I)\\ &= i\,\mathrm{Im}\,\Phi_{\{0\}}(G) + \int_0^{2\pi} \frac{e^{-it}+\lambda}{e^{-it}-\lambda}\,d\mu(e^{it})(G), \qquad \lambda \in \mathbb{D}.\end{aligned}$$

In particular, the identities

$$\begin{cases}\Phi_{\{0\}} = i\,\mathrm{Im}\,\mathscr{I}^*\Phi_{\{0\}}\mathscr{I} + \int_0^{2\pi} d\mathscr{I}^*\mu(e^{it})\,\mathscr{I}\\ \Phi_{\{n\}}Z^n = 2\int_0^{2\pi} e^{int}\,d\mathscr{I}^*\mu(e^{it})\,\mathscr{I} \qquad \text{for} \quad n \geqslant 1,\end{cases} \tag{7.25}$$

holds as operator on $\ell^2_{\mathscr{N}}$.

The time domain realization (7.25) of Φ implies that $d\mu$ admits a Fourier expansion

$$d\mu(e^{it}) \overset{w}{=} \sum_{j=-\infty}^{\infty} e^{-ijt}\,d\mu_j(e^{it}),$$

where for $j \geqslant 0$, $\mathscr{I}^*\,d\mu_j JZ^{-j}$ are measures with values in $\mathscr{D}$. Conversely, given a measure $d\mu$ of this form and a selfadjoint operator $A \in \mathscr{D}$, then with

$$\begin{cases}\Psi_{\{0\}} = i\mathscr{I}^*A\mathscr{I} + \int_0^{2\pi} d\mathscr{I}^*\mu(e^{it})\,\mathscr{I}\\ \Psi_{\{n\}} = 2\int_0^{2\pi} e^{int}\,d\mathscr{I}^*\mu(e^{it})\,\mathscr{I}Z^{-n} \qquad \text{for} \quad n \geqslant 1,\end{cases}$$

the series $\theta(\lambda) = \sum_{j=0}^{\infty} \lambda^j \Psi_{\{j\}}$ converges weakly to

$$iA + \int_0^{2\pi} \frac{e^{-it}+\lambda}{e^{-it}-\lambda}\,d\mu(e^{it}).$$

The real part of θ is nonnegative, but without further conditions it is not clear that θ is the characteristic function of a Carathéodory operator in $\mathscr{U}$.

Proof of Theorem 7.4. Since α_Φ is unitary it has a unique resolution of the identity $\{E(t)\}_{t \in \mathbb{R}}$ supported by $[0, 2\pi]$ such that

$$\alpha_\Phi = \int_0^{2\pi} e^{it}\,dE(t). \tag{7.26}$$

Plugging (7.26) into (7.23) and using functional calculus we get

$$(\Phi G)^{\triangle}\,(\lambda I) = i\,\mathrm{Im}\,\delta_\Phi(G) + \frac{1}{2}\int_0^{2\pi} \frac{e^{-it}+\lambda}{e^{-it}-\lambda}\,d\gamma_\Phi E(t)\,\gamma_\Phi^*(G), \qquad \lambda \in \mathbb{D}.$$

This proves the first part of the theorem with $\mu(e^{it}) = \frac{1}{2}\gamma_{\Phi} E(t)\, \gamma_{\Phi}^{*}$. Comparing the power series in λ on both sides of this equation we obtain

$$\begin{cases} \Phi_{\{0\}}\, G = i \operatorname{Im} \delta_{\Phi}(G) + \int_0^{2\pi} d\mu(e^{it})(G) \\ \Phi_{\{n\}}\, Z^n G Z^{*n} = 2 \int_0^{2\pi} e^{int}\, d\mu(e^{it})(G) \qquad \text{for} \quad n \geqslant 1. \end{cases}$$

Let $g \in \ell^2_{\mathcal{N}}$ and set $G = \mathcal{J}(g)$. Then

$$\mathcal{J}^*(\Phi_{\{0\}}\, G) = \Phi_{\{0\}}\, g = \mathcal{J}^* \left(i \operatorname{Im} \delta_{\Phi} + \int_0^{2\pi} d\mu(e^{it}) \right) \mathcal{J}(g)$$

and

$$\mathcal{J}^*(\Phi_{\{n\}}\, Z^n G Z^{*n}) = \Phi_{\{n\}}\, Z^n g = \mathcal{J}^* \left(2 \int_0^{2\pi} e^{int}\, d\mu(e^{it}) \right) \mathcal{J}(g),$$

and this implies (7.25). ∎

THEOREM 7.5. *Let $\Phi \in \mathcal{U}$ be a Carathéodory function. Then there exists a finite nonnegative Borel measure $dm(e^{it})$ with values in $\mathcal{L}(\ell^2_{\mathcal{N}})$ such that for all $c, d \in \mathcal{N}$,*

$$\langle \Phi_{k-n,k}\, c, d \rangle_{\mathcal{N}}$$
$$= \begin{cases} \langle i \operatorname{Im} \mathcal{J}^* \Phi_{\{0\}}\, \mathcal{J}(e_k c), e_k d \rangle_{\ell^2_{\mathcal{N}}} + \int_0^{2\pi} d\langle m(e^{it})(e_k c), e_k d \rangle_{\ell^2_{\mathcal{N}}}, \\ \qquad n = 0, \quad k \in \mathbb{Z}, \\ 2 \int_0^{2\pi} d\langle m(e^{it})(e^{ikt} e_k c), e^{i(k-n)\,t} e_{k-n} d \rangle_{\ell^2_{\mathcal{N}}}, \\ \qquad n \geqslant 1, \quad k \in \mathbb{Z}, \end{cases} \tag{7.27}$$

where $e_k \in \mathcal{L}(\mathcal{N}; \ell^2_{\mathcal{N}})$ is defined by $(..., 0, \boxed{0}, 0, ..., 0, I_{\mathcal{N}}, 0, ...)^t \in \ell^2_{\mathcal{N}}$ (here $I_{\mathcal{N}}$ is placed in the kth coordinate).

If we introduce the stochastic process $x(k) = e^{ikt} e_k$, then the formula (7.27) can be written as

$$\langle \Phi_{k-n,k}\, c, d \rangle_{\mathcal{N}}$$
$$= \begin{cases} \langle i \operatorname{Im} \mathcal{J}^* \delta_{\Phi} \mathcal{J}(x(0)\, c), x(0)\, d \rangle_{\ell^2_{\mathcal{N}}} + \langle x(0)\, c, x(0)\, d \rangle_{\mathbf{L}^2_{dm}([0,2\pi];\, \ell^2_{\mathcal{N}})}, \\ \qquad n = 0, \quad k \in \mathbb{Z}, \\ 2\langle x(k)\, c, x(k-n)\, d \rangle_{\mathbf{L}^2_{dm}([0,2\pi];\, \ell^2_{\mathcal{N}})}, \\ \qquad n \geqslant 1, \quad k \in \mathbb{Z}, \end{cases}$$

where $\mathbf{L}^2_{dm}([0, 2\pi]; \ell^2_{\mathcal{N}})$ is the space of all $\ell^2_{\mathcal{N}}$-valued functions on $\mathbb{T}$ which are square summable with respect to m.

Proof of Theorem 7.5. Let $dm(e^{it})\colon \ell^2_{\mathcal{N}} \to \ell^2_{\mathcal{N}}$ be the measure defined by $dm(e^{it}) = \mathcal{J}^*\, d\mu(e^{it})\, \mathcal{J}$. Then by the previous theorem, we have

$$\begin{cases} \Phi_{\{0\}} = i \operatorname{Im} \mathcal{J}^* \delta_{\Phi} \mathcal{J} + \int_0^{2\pi} dm(e^{it}) \\ \Phi_{\{n\}} Z^n = 2 \int_0^{2\pi} e^{int}\, dm(e^{it}) \qquad \text{for} \quad n \geqslant 1. \end{cases} \tag{7.28}$$

Thus

$$\begin{aligned} \langle \Phi_{k,k} c, d \rangle_{\mathcal{N}} &= \langle \Phi_{\{0\}} e_k c, e_k d \rangle_{\ell^2_{\mathcal{N}}} \\ &= \langle i \operatorname{Im} \mathcal{J}^* \delta_{\Phi} \mathcal{J}(e_k c), e_k d \rangle_{\ell^2_{\mathcal{N}}} + \int_0^{2\pi} d\langle m(e^{it})(e_k c), e_k d \rangle_{\ell^2_{\mathcal{N}}}. \end{aligned}$$

Since

$$\langle \Phi_{\{n\}} Z^n e_k c, e_{k-n} d \rangle_{\ell^2_{\mathcal{N}}} = \langle \Phi_{\{n\}} e_{k-n} c, e_{k-n} d \rangle_{\ell^2_{\mathcal{N}}} = \langle \Phi_{k-n,k} c, d \rangle_{\mathcal{N}}$$

and

$$\begin{aligned} \langle \Phi_{\{n\}} Z^n e_k c, e_{k-n} d \rangle_{\ell^2_{\mathcal{N}}} &= 2 \int_0^{2\pi} d e^{int} \langle m(e^{it})(e_k c), e_{k-n} d \rangle_{\ell^2_{\mathcal{N}}} \\ &= 2 \int_0^{2\pi} d\langle m(e^{it})(e^{ikt} e_k c), e^{i(k-n)t} e_{k-n} d \rangle_{\ell^2_{\mathcal{N}}}, \end{aligned}$$

formula (7.27) holds. ∎

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